RNC WORKSHOP
BOUNDARY VALUES OF MAPPINGS OF FINITE DISTORTION

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ABSTRACT. We find a sufficient condition for a weakly differentiable homeomorphism in Euclidean space to have a homeomorphic extension to the boundary of its domain of definition. In a certain sense, this condition is best possible.

1. INTRODUCTION

A classical theorem of Carathéodory [3] and Osgood and Taylor [22] asserts that a conformal mapping between planar Jordan domains extends to a homeomorphism of their closures. This result can be generalized to quasiconformal mappings in the plane, see [19, p.42]. On the other hand, Kuusalo [18] showed that for any $n \geq 3$ there exists a Jordan domain $\Omega \subset \mathbb{R}^n$ and a quasiconformal bijection $f : \Omega \to \Omega$ that does not have a continuous boundary extension. Therefore, higher-dimensional versions of the Carathéodory extension theorem must include additional assumptions on the domains in question.

Väisälä [25] established an analog of Carathéodory's theorem for Jordan domains that are quasiconformally equivalent to a ball. Later, he and Näkki introduced local conditions on the boundaries of domains Ω and Ω' that guarantee that every quasiconformal map $f : \Omega \to \Omega'$ extends to a given boundary point. In many situations these boundary properties (called P_1 and P_2 in [26] and quasiconformal flatness and accessibility in [21]) turn out to be the optimal assumptions, see [11].

Recently, Ryazanov, Srebro and Yakubov [23] generalized the Carathéodory theorem in a different direction by proving that every *BMO-qc* homeomorphism of planar Jordan domains extends to a homeomorphism of their closures. Later, same authors and Martio [20] proved the extension property for BMO-qc automorphisms of a half-space in $\mathbb{R}^n$. (See also [4]). Recall that for a quasiconformal mapping f in $\mathbb{R}^n$ the *distortion function* $K_O(x, f) = |Df(x)|^n/J(x, f)$ is bounded a.e. (Here $|Df(x)|$ is the operator norm of the differential matrix $Df(x)$ and $J(x, f) = \det Df(x)$.) In contrast, a BMO-qc homeomorphism in general has an unbounded distortion function that is majorized a.e. by a function in the class BMO. By [12, Theorem 18.3.2] this condition is equivalent (on bounded domains) to the integrability of $\exp(\lambda K_O(\cdot, f))$ for some $\lambda > 0$.

In the present paper we establish the sharp Orlicz-type condition on $K_O(\cdot, f)$ under which Carathéodory-type results still hold. This condition, usually called *subexponential integrability*, was initially introduced in [15]; see section 2 for its precise formulation. We also prove a result concerning extension of the inverse mapping; note that in contrast to the quasiconformal case, the inverse of a mapping with subexponentially integrable distortion may not belong to the same class. It turns out that f^{-1} can be continuously extended to the boundary under somewhat weaker assumptions on f, see Corollary 2. Here we state a simplified version of our results.

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Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a Jordan domain, and let $\Omega' \subset \mathbb{R}^n$ be a bounded domain that is quasiconformally equivalent to a ball. Then every homeomorphism $f : \Omega \to \Omega'$ with subexponentially integrable distortion extends to a homeomorphism $\bar{f} : \overline{\Omega} \to \overline{\Omega'}$.*

The precise statements and proofs are in Section 3. Their sharpness is established in Section 4. In Section 5 we consider mappings of finite distortion from the halfspace onto itself. More specifically, we generalize the Riemann-Schwarz reflection principle and the Ahlfors-Beurling quasisymmetry theorem.

2. Preliminaries

Throughout the paper Ω and Ω' are domains (i.e. nonempty connected open sets) in $\mathbb{R}^n$. A mapping $f \in W^{1,1}(\Omega; \mathbb{R}^n)$ is called *a mapping of finite distortion* if its Jacobian determinant $J(\cdot, f)$ is in $L^1(\Omega)$ and $J(x, f) > 0$ for a.e. $x \in \Omega$ such that $|Df(x)| \neq 0$. Define the outer distortion K_O of f by $K_O(x, f) = |Df(x)|^n / J(x, f)$ when $|Df(x)| \neq 0$ and $K_O(x, f) = 1$ otherwise. The function $K_O(\cdot, f)$ is *subexponentially integrable* if there exists an Orlicz function $\mathcal{A}$ (i.e. an increasing C^∞-diffeomorphism $\mathcal{A} : (0, \infty) \to (0, \infty)$) such that

$$\int_1^\infty \frac{\mathcal{A}'(t)}{t}\, dt = \infty, \tag{1}$$

$$\exists t_0 \in (0, \infty) \text{ such that } t\mathcal{A}'(t) \text{ increases to infinity on } [t_0, \infty), \tag{2}$$

and $\int_\Omega \exp(\mathcal{A}(K_O(x, f)))dx < \infty$. Note that in the literature the above integrability conditions on $|Df|$ and $K_O(\cdot, f)$ are often stated in their local forms. Naturally, we need global integrability in order to study the boundary behavior of f.

Of course, the condition $\int_\Omega \exp(\mathcal{A}(K_O(x, f)))dx < \infty$ forces Ω to have a finite Lebesgue measure. However, this does not mean that our results are not applicable to mappings in domains of infinite measure. As in the proof of Theorem 6 below, one can infer the existence of boundary values by considering appropriate bounded subdomains.

Let $f : \Omega \to \mathbb{R}^n$ be a mapping with subexponentially integrable outer distortion. By (2) we have $\lim_{t\to\infty} \mathcal{A}(t)/\log t = \infty$, which in turn implies that $\exp(\mathcal{A}(t))$ dominates t^p for every $p \in [1, \infty)$. Therefore, $K_O(\cdot, f) \in L^p(\Omega)$ for $1 \leq p < \infty$. By [15, Theorem 1.1], f is continuous in Ω and is either constant or both open and discrete. The proof of Theorem 1.1 [15] also demonstrates that $\int_\Omega P(|Df(x)|^n)dx < \infty$ for some Orlicz function P such that

$$\int_1^\infty \frac{P(t)}{t^2} dt = \infty, \tag{3}$$

and

$$\exists t_0 \in (0, \infty) \text{ such that } t \mapsto t^{-n/(n+1)} P(t) \text{ is increasing on } (t_0, \infty). \tag{4}$$

Now we present some metric and topological definitions, most of which can be found in [11, 21, 26]. Given $\Omega \subset \mathbb{R}^n$, a measurable function $\omega : \Omega \to [0, \infty]$ and a path family Γ in Ω, define *the weighted modulus* of Γ by

$$\operatorname{mod}_\omega(\Gamma) = \inf \left\{ \int_\Omega \rho(x)^n \omega(x)\, dx \right\},$$

where the infimum is taken over all Borel functions $\rho : \Omega \to [0, \infty)$ such that $\int_\gamma \rho \geq 1$ for each $\gamma \in \Gamma$. If E and F are subsets of Ω, we write

$$\mathrm{mod}_\omega(E, F; \Omega) = \mathrm{mod}_\omega(\Gamma_{E,F}),$$

where $\Gamma_{E,F}$ is the family of all paths connecting E and F in Ω. The subscript ω is dropped when $\omega \equiv 1$.

We use notations $\mathbb{B}(a, r) = \{x \in \mathbb{R}^n : |x - a| < r\}$ and $\mathbb{B} = \mathbb{B}(0, 1)$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. Given a mapping $f : \Omega \to \mathbb{R}^n$ and a point $b \in \partial\Omega$, define *the cluster set* of f at b by $C(f, b) = \bigcap_{r>0} \overline{f(\mathbb{B}(b, r) \cap \Omega)}$.

The following definition introduces a few concepts related to domains in the Euclidean space. Here and in the sequel the topological terms (open set, closure, neighborhood, etc.) are understood in the sense of the topology of $\mathbb{R}^n$, unless noted otherwise.

Definition. A domain Ω is

(a) *a Jordan domain* if $\partial\Omega$ is homeomorphic to the $(n - 1)$-dimensional sphere S^{n-1}.

(b) *locally connected* at $b \in \partial\Omega$ if for any $r > 0$ there exists an open set $U \subset \mathbb{B}(b, r)$ such that $b \in U$ and $U \cap \Omega$ is connected.

(c) *finitely connected* at $b \in \partial\Omega$ if for any $r > 0$ there exists an open set $U \subset \mathbb{B}(b, r)$ such that $b \in U$ and $U \cap \Omega$ consists of a finite number of connected components.

(d) *QC flat* at $b \in \partial\Omega$ if for any subdomains $E, F \subset \Omega$ one has $\mathrm{mod}(E, F; \Omega) = \infty$ whenever $b \in \overline{E} \cap \overline{F}$.

(e) *QC accessible* at $b \in \partial\Omega$ if for any neighborhood U of b there is a continuum $E \subset \Omega$ and a constant $\delta > 0$ such that $\mathrm{mod}(E, F; \Omega) \geq \delta$ for every subdomain $F \subset \Omega$ that meets ∂U and has point b on its boundary.

Note that the term "QC flatness" is often used (e.g. [6, 7, 28]) to mean a different property which implies both QC flatness and QC accessibility. Any Jordan domain is locally connected at every point of its boundary [27, p.66]. A planar Jordan domain is also QC flat and QC accessible at every boundary point, as follows from the Riemann mapping theorem and the Carathéodory extension theorem. In general, $\Omega \subset \mathbb{R}^n$ is both QC flat and QC accessible at every boundary point if any one of the following conditions is satisfied: (a) $\partial\Omega$ is a C^1 manifold [26, Theorem 17.12]; (b) Ω is locally quasiconformally collared [26, Theorem 17.10]; (c) Ω is a quasiextremal distance domain [11, Lemma 3.1]; (d) Ω is a uniform domain [10, Lemma 2.18].

3. Existence of a continuous boundary extension

Theorem 2. *Let Ω and Ω' be bounded domains in $\mathbb{R}^n$, and let $f : \Omega \to \Omega'$ be a homeomorphism of finite distortion. Suppose that $\int_\Omega \exp(\mathcal{A}(K_O(x, f)))dx < \infty$ for some Orlicz function $\mathcal{A}$ that satisfies* (1) *and* (2).

If Ω is locally connected at $b \in \partial\Omega$ and if some point of $C(f, b)$ is QC accessible for Ω', then $\lim_{x\to b} f(x)$ exists.

Proof. Since f is a homeomorphism, it follows that $C(f, b) \subset \partial\Omega'$. Let $b' \in C(f, b)$ be a QC accessible point for Ω'. Suppose that $C(f, b)$ contains another point, say b''. Let V be a neighborhood of b' such that $b'' \notin \overline{V}$, and let $E \subset \Omega'$ be a continuum whose existence is guaranteed by the definition of QC accessibility.

Choose a sequence of neighborhoods U_j of b such that $\mathrm{diam}(U_j) \to 0$ and $U_j \cap \Omega$ is connected for each j. By [17, Theorem 5.3] we have $\mathrm{mod}_{K_O(\cdot,f)^{n-1}}(U_j, f^{-1}(E); \Omega) \to 0$.

Using [14, Lemma 3.2] and [16, Theorem 1.2], we conclude that the assumptions of Theorem 4.1 [17] are fulfilled. The latter theorem yields

$$\operatorname{mod}(f(U_j), E; \Omega') \le \operatorname{mod}_{K_O(\cdot,f)^{n-1}}(U_j, f^{-1}(E); \Omega),$$

which implies that $\operatorname{mod}(f(U_j), E; \Omega') \to 0$ as $j \to \infty$. On the other hand, $f(U_j)$ is a subdomain of Ω' such that $b', b'' \in \overline{f(U_j)}$, hence $f(U_j) \cap \partial V \neq \varnothing$. This contradicts the assumption that b' is QC accessible. Therefore, $C(f, b)$ contains only one point. □

Corollary 1. *Let Ω, Ω' and f be as in Theorem 2. Suppose in addition that Ω is locally connected on the whole boundary and Ω' is QC accessible at every boundary point. Then f has a continuous extension to $\overline{\Omega}$.*

Proof. Define $f(b) = \lim_{x\to b} f(x)$ for every boundary point b. We only have to check that $\lim_{n\to\infty} f(b_n) = f(b)$ for any sequence $\{b_n\} \subset \partial\Omega$ converging to b. This is done by a standard argument: for every n choose $x_n \in \Omega$ such that $|x_n - b_n| < 1/n$ and $|f(x_n) - f(b_n)| < 1/n$. Since $x_n \to b$, it follows that $f(x_n) \to f(b)$, which it turn implies $f(b_n) \to f(b)$. □

The next theorem concerns the existence of the continuous boundary extension of f^{-1}. In order to state it in a precise form, we need to define the inner distortion function $K_I(\cdot, f)$. Let $D^\sharp f(x)$ be the cofactor matrix of $Df(x)$. Define $K_I(x, f) = |D^\sharp f(x)|^n / J(x, f)^{n-1}$ if $|D^\sharp f(x)| \neq 0$ and $K_I(x, f) = 1$ otherwise. It is easy to see that $K_I(x, f) \le K_O(x, f)^{n-1}$ and $K_O(x, f) \le K_I(x, f)^{n-1}$ (e.g. [12, section 6.4]).

Theorem 3. *Let Ω and Ω' be bounded domains in $\mathbb{R}^n$, and let $f : \Omega \to \Omega'$ be a homeomorphism of finite distortion. Suppose that $\int_\Omega K_I(x, f)dx < \infty$ and there exists an Orlicz function P such that (3) and (4) are satisfied and $\int_\Omega P(|Df(x)|^n)dx < \infty$.*

If Ω is finitely connected at two distinct points $b_1, b_2 \in \partial\Omega$, then Ω' is not QC flat at any point of $C(f, b_1) \cap C(f, b_2)$.

By virtue of the results stated in Section 2, the analytic assumptions of Theorem 3 hold for every mapping that satisfies the assumptions of Theorem 2.

Proof. For $j = 1, 2$ let U_j be a neighborhood of b_j such that $U_j \cap \Omega$ has finitely many components and $d = \operatorname{dist}(U_1, U_2) > 0$. Now if $b' \in C(f, b_1) \cap C(f, b_2)$, then $b' \in \overline{f(U_1 \cap \Omega)} \cap \overline{f(U_2 \cap \Omega)}$. Let V_j be a connected component of $U_j \cap \Omega$ with the property that $b' \in \overline{f(V_j)}$. We have

$$\operatorname{mod}_{K_I(\cdot,f)}(V_1, V_2; \Omega) \le \int_\Omega d^{-n} K_I(x, f)\, dx < \infty,$$

since we can take $\rho(x) \equiv 1/d$ in the definition of the weighted modulus.

As in the above proof of Theorem 2, we can apply Theorem 4.1 [17] to obtain

$$\operatorname{mod}(f(V_1), f(V_2); \Omega') \le \operatorname{mod}_{K_I(\cdot,f)}(V_1, V_2; \Omega) < \infty.$$

Therefore, Ω' is not QC flat at b'. □

Corollary 2. *Let Ω, Ω' and f be as in Theorem 3. Suppose in addition that Ω is finitely connected on the whole boundary and Ω' is QC flat at every boundary point. Then f^{-1} has a continuous extension to $\overline{\Omega'}$.*

Proof. Consider a boundary point $b' \in \partial\Omega'$. If the cluster set $C(f^{-1}, b')$ contains two distinct points b_1, b_2, then $b' \in C(f, b_1) \cap C(f, b_2)$, which in view of Theorem 3 contradicts the QC flatness at b'. Therefore, f^{-1} has a limit at b'. As in Corollary 1, we see that f^{-1} is continuous in $\overline{\Omega'}$. □

Corollary 3. *Let Ω, Ω' and f be as in Theorem 2. Suppose in addition that Ω is locally connected on the whole boundary and Ω' is QC accessible and QC flat at every boundary point. Then f extends to a homeomorphism of $\overline{\Omega}$ onto $\overline{\Omega'}$.*

4. Sharpness of the integrability conditions

Our first example shows that the assumption on the size of $K_O(\cdot, f)$ in Theorem 2 and Corollary 1 cannot be weakened.

Theorem 4. *Suppose that $\mathcal{A}$ is an Orlicz function such that*

$$\int_1^\infty \frac{\mathcal{A}'(t)}{t} < \infty.$$

Then there exist domains Ω and Ω' satisfying the assumptions of Corollary 1 and a homeomorphism of finite distortion $f : \Omega \to \Omega'$ such that

$$\int_\Omega \exp(\mathcal{A}(K_O(x, f)))dx < \infty, \tag{5}$$

but f has no continuous extension to $\overline{\Omega}$.

Proof. Following [15], define

$$\rho(t) = \exp\left(3^n \int_0^t \frac{ds}{s\mathcal{A}^{-1}(\log \frac{e}{s})}\right), \quad 0 \le t \le 1.$$

Consider the sets

$$\Omega = \{x \in \mathbb{B} : x_n < 0\}$$

and

$$\Omega' = \{x \in \mathbb{B}(0, \rho(1)) \setminus \overline{\mathbb{B}} : x_n < 0\}.$$

Clearly, Ω is locally connected on the boundary and Ω' is QC accessible at every boundary point (see [26], Remark 17.24(3)).

Define $f : \Omega \to \Omega'$ by the rule

$$f(x) = \frac{x}{|x|}\rho(|x|).$$

Thus f will map homeomorphically the set Ω onto Ω'. Following the proof of Theorem 1.2 in [15] we find that f is a mapping of finite distortion and (5) holds. Since $\rho(0) = 1$, it follows that the cluster set of f at the origin is the half-sphere $\{x : |x| = 1, x_n \le 0\}$. Thus f cannot be continuously extended to the origin. □

Next we demonstrate the sharpness of conditions in Theorem 3 and Corollary 2.

Theorem 5. *There exist domains Ω and Ω' and a homeomorphism $f : \Omega \to \Omega'$ of finite distortion such that Ω and Ω' satisfy the assumptions of Corollary 2, f is Lipschitz continuous and*

$$\int_\Omega (K_I(x, f))^q \, dx < \infty$$

for all $0 < q < 1$, but f^{-1} has no continuous extension to $\overline{\Omega'}$.

Proof. We set $s(x) = \sqrt{x_1^2 + x_2^2 + ... + x_{n-1}^2}$ and consider the sets

$$\Omega = \Omega' = \{x \in \mathbb{R}^n : x_1 > 0, s(x) < 1 \text{ and } |x_n| < 2\}.$$

Since a half-cylinder is bilipschitz equivalent to a ball, it follows that Ω and Ω' satisfy the assumptions of Corollary 2.

Following [2], define $f : \Omega \to \Omega'$ by the rule

$$f(x) = \begin{cases} (x_1, x_2, ..., x_{n-1}, s(x)x_n) & \text{for } |x_n| < 1; \\ (x_1, x_2, ..., x_{n-1}, [2(|x_n| - 1) + (2 - |x_n|)s(x)] \operatorname{sgn} x_n) & \text{for } 1 \leq |x_n| \leq 2. \end{cases}$$

The construction of Example 1 in [2] shows that f is Lipschitz and $J(x, f) > 0$ for every $x \in \Omega$. So f is a mapping of finite distortion and a direct computation shows that

$$\int_\Omega (K_I(x, f))^q \, dx \leq C_n \int_\Omega s(x)^{-q(n-1)} dx < \infty.$$

On the other hand, $f(x) \to 0$ whenever $s(x) \to 0$ and $|x_n| \leq 1$, which implies that $C(f^{-1}, 0) = \{x : x_1 = \cdots = x_{n-1} = 0, |x_n| \leq 1\}$. Therefore, f^{-1} cannot be continuously extended to the boundary of Ω'. □

5. Self-maps of the halfspace

The Riemann-Schwarz reflection principle for conformal mappings has been extended by Väisälä to the quasiconformal case [26, Theorem 35.2]. Recently it has been generalized to BMO-qc mappings in [20, 23]. Using the results in Section 3 we can further weaken the restrictions concerning the distortion function of a mapping.

Let $\mathbb{H}^n_\pm = \{x \in \mathbb{R}^n : \pm x_n > 0\}$ be the halfspaces separated by the hyperplane $\mathbb{P} = \{x \in \mathbb{R}^n : x_n = 0\}$. The reflection in $\mathbb{P}$ is given by $S(x_1, \dots, x_n) = (x_1, \dots, -x_n)$.

Theorem 6. *Let Ω and Ω' be subdomains of $\mathbb{H}^n_+$. Suppose that the sets $E = \partial\Omega \cap \mathbb{P}$ and $E' = \partial\Omega' \cap \mathbb{P}$ are relatively open in $\partial\Omega$ and $\partial\Omega'$ respectively. Let $f : \Omega \to \Omega'$ be a homeomorphism such that $\bigcup_{x \in E} C(f, x) = E'$. Suppose further that the restriction of f to every bounded subdomain of Ω is a mapping of subexponentially integrable distortion.*

Then f extends to a homeomorphism from $D = \Omega \cup E \cup S(\Omega)$ onto $D' = \Omega' \cup E' \cup S(\Omega')$. The extended mapping also has subexponentially integrable distortion on bounded subdomains.

Proof. It is easy to see that Ω is locally connected at every point of E, and Ω' is both QC flat and QC accessible at every point of E'. By Theorem 2 the mapping f has a continuous extension $F : \Omega \cup E \to \Omega' \cup E'$, which is a homeomorphism by virtue of Theorem 3. For $x \in S(\Omega)$ we define $F(x) = S(f(S(x)))$, thus making F into a homeomorphism of D onto D'.

Let G be a bounded subdomain of D that is symmetric about $\mathbb{P}$. Since $f \in W^{1,1}(G \cap \mathbb{H}^n_+)$, it follows that f is ACL in $G \cap \mathbb{H}^n_+$ [5, Theorem 4.9.2]. Therefore, F is ACL in G, and the same theorem yields $F \in W^{1,1}(G)$. It remains to observe that $K_O(\cdot, F)$ is a.e. equal to the symmetric extension of $K_O(\cdot, f)$. □

The idea of the proof of Theorem 4 can be used to show that the subexponential integrability assumption in Theorem 6 cannot be weakened.

Finally we present a generalization of the classical theorem of Ahlfors and Beurling that says that the trace of a quasiconformal automorphism of $\mathbb{H}^2_+$ is quasisymmetric [1, p.65]. It should be noted that Ahlfors and Beurling also proved that every quasisymmetric homeomorphism of $\mathbb{R}$ extends to a quasiconformal automorphism of $\mathbb{H}^2_+$. It would be interesting to find a necessary and sufficient condition for a homeomorphism of $\mathbb{R}$ to be the trace of a mapping with (locally) subexponentially integrable distortion. For results in this direction, see [4, 24].

Theorem 7. *In addition to the assumptions of Theorem 6, assume that $n = 2$ and $\Omega = \Omega' = \mathbb{H}^2_+$. Then for each $s, t \in \mathbb{R}$*

$$M^{-1} \leq \frac{F(s+t) - F(s)}{F(s) - F(s-t)} \leq M,$$

where $M = A \exp\left(Bt^{-2} \int_{\mathbb{B}(s,2t)\cap\mathbb{H}^2_+} K_O(x,f)\,dx\right)$ with absolute constants A and B.

Proof. This is an immediate generalization of Theorem 2.1 in [24], and it can be proved in exactly the same way with the help of Theorem 4.1 [17]. □

For $n > 2$, Gehring [8, 9] proved that a quasiconformal automorphism of the half-space $\mathbb{H}^n_+$ induces a quasiconformal automorphism of its boundary $\mathbb{P}$. This follows from the fact (also established by Gehring) that quasiconformal mappings admit an equivalent metric definition. However, it seems difficult, if not impossible, to find a metric definition of mappings of finite (in particular subexponentially integrable) distortion [13]. This presents an obstacle to obtaining an analog of Gehring's result for the mappings of finite distortion.

Indeed, consider a homeomorphism $f : \mathbb{H}^n_+ \to \mathbb{H}^n_+$ such that $K_O(\cdot, f)$ is subexponentially integrable on bounded subdomains of $\mathbb{H}^n_+$. By Theorem 6 f induces a homeomorphism $F : \mathbb{P} \to \mathbb{P}$. If $K_O(\cdot, f)$ is bounded in a neighborhood of every point of $\mathbb{P}$, then F can be shown to be a mapping of finite distortion. However, no integrability assumption on $K_O(\cdot, f)$ can exclude the possibility that $\lim_{x\to b} K_O(x, f) = \infty$ for every $b \in \mathbb{P}$. At present it remains unclear which regularity properties are possessed by the induced boundary mapping.

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