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Hyperbolic and quasisymmetric structure of hyperspaces

Leonid V. Kovalev and Jeremy T. Tyson

Abstract. A hyperspace is a space of nonempty closed sets equipped with the Hausdorff metric. Among the subjects considered in this paper are Gromov hyperbolicity, quasisymmetric equivalence and bi-Lipschitz embeddings of hyperspaces.

1. Introduction

Let (X, d) be a metric space. We denote by $\mathrm{CL}(X)$ the hyperspace of nonempty closed subsets of a metric space X, equipped with the Hausdorff distance

$$D(A, B) = \inf\{\epsilon > 0 : A \subset N_\epsilon(B) \text{ and } B \subset N_\epsilon(A)\},$$

where $N_\epsilon(A) = \{x \in X : \mathrm{dist}(x, A) \le \epsilon\}$ is the closed ϵ-neighborhood of $A \subset X$. If there is no such ϵ, the distance between A and B is infinite. Notice that if $D(A, B) < \infty$, then the above infimum is attained. The distance D is a metric on $\mathrm{CL}(X)$ if and only if X is bounded. For unbounded X the set $\mathrm{CL}(X)$ naturally splits into an infinite collection of disjoint metric spaces. The most notable of these is $\mathcal{H}(X)$, the space of nonempty bounded closed subsets of X. However, we are also interested in other components of $\mathrm{CL}(X)$. Each of them has the form $\mathcal{H}(X; C) := \{A \in \mathrm{CL}(X) : D(A, C) < \infty\}$ for some $C \in \mathrm{CL}(X)$. Note that $\mathcal{H}(X; C) = \mathcal{H}(X)$ if and only if C is bounded. Also, let $\mathcal{H}^c(X; C) = \{A \in \mathcal{H}(X; C) : A \text{ is geodesically convex}\}$ denote the hyperspace of closed convex sets at a finite distance from C. When C is bounded, we write $\mathcal{H}^c(X)$ for $\mathcal{H}^c(X; C)$. An open ball with center x and radius r will be denoted by $B(x, r)$.

Most of the existing research on hyperspaces equipped with the Hausdorff distance is focused on $\mathcal{H}(X)$ and its subsets such as $\mathcal{H}^c(X)$. However, the spaces $\mathcal{H}(X; C)$ with C unbounded do arise naturally. As an example, consider a Lipschitz quotient mapping $f : X \to Y$, where X and Y are metric spaces. Recall that f is called a Lipschitz quotient [**4**] if there exist two constants $0 < l \le L < \infty$ such that

$$B(f(x), lr) \subset f(B(x, r)) \subset B(f(x), Lr), \quad x \in X, \ r > 0.$$

A Lipschitz quotient $f : X \to Y$ induces a map $G_f : \mathrm{CL}(Y) \to \mathrm{CL}(X)$ such that $G_f(A) = f^{-1}(A)$. It is easy to check that G_f is bi-Lipschitz on each component

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of $\mathrm{CL}(Y)$. However, in many interesting cases G_f does not map $\mathcal{H}(Y)$ into $\mathcal{H}(X)$; instead $G_f(\mathcal{H}(Y))$ is contained in $\mathcal{H}(X;C)$ for some unbounded set C. Even the simplest elements of $\mathcal{H}(Y)$, namely singletons $\{y\}$, can be mapped by G_f into sets with rather complicated structure [**12, 28**]. It remains an open question whether $G_f(\{y\})$ is always a discrete set when $X = Y = \mathbb{R}^n$, $n > 2$. See [**4, 20, 23**] for more on this problem.

The spaces $\mathcal{H}(X;C)$ with C unbounded are typically larger and more complex than $\mathcal{H}(X)$. For instance, $\mathcal{H}(X)$ is separable whenever X is *proper* (i.e. such that all of its closed bounded subsets are compact). By contrast, $\mathcal{H}(X;C)$ is non-separable if X is connected and C is unbounded (Theorem 5.1). Furthermore, large classes of metric spaces admit isometric or bi-Lipschitz embeddings into $\mathcal{H}(X;C)$.

Section 2 is concerned with hyperspaces of special classes of metric spaces: length spaces, Gromov hyperbolic space, and nonpositively curved spaces (in the sense of Busemann or Cartan–Alexandrov–Toponogov). We attempt to determine the extent to which such properties of X are inherited by $\mathcal{H}(X)$ or other components of $\mathrm{CL}(X)$. Several results in section 2 are extensions to the case of general hyperspaces $\mathcal{H}(X;C)$ of known results for the standard hyperspace $\mathcal{H}(X)$ from [**11**] and [**13**]. Sections 3 and 4 are focused on mappings between hyperspaces. The following class of mappings is central in modern geometric function theory [**16**].

Definition 1.1. Let X and Y be metric spaces. An injective mapping $f : X \to Y$ is called *quasisymmetric*, or η-quasisymmetric, if there is a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that

$$\frac{d_Y(f(x_1), f(x_3))}{d_Y(f(x_2), f(x_3))} \le \eta\left(\frac{d_X(x_1,x_3)}{d_X(x_2,x_3)}\right)$$

for any distinct points $x_1, x_2, x_3 \in X$.

In general, a quasisymmetric mapping $f : X \to Y$ does not lift to a quasisymmetric mapping of $\mathcal{H}(X)$. We call f *hyperquasisymmetric* if it does. Theorem 3.4 provides two characterizations of such maps, which arise naturally in the context of Gromov hyperbolicity and bi-Lipschitz homogeneity. In §4 we consider a notion of dimension of metric spaces which is invariant under hyperquasisymmetric mappings.

2. Geodesics in hyperspaces

Throughout the paper all rectifiable curves are assumed to be parametrized proportional to the arclength, unless stated otherwise. A metric space (X,d) is called a *length space* if for any $x, y \in X$ and any $\epsilon > 0$ there is a curve $\gamma : [0,1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = y$ and the length of γ is at most $d(x,y) + \epsilon$. If such γ exists even for $\epsilon = 0$, then X is called a *geodesic space*. Every proper length space is geodesic [**15, 29**].

The hyperspace $\mathcal{H}(X)$ of a geodesic space X need not be geodesic itself, as is demonstrated by the following example from [**10**]. Let X be the Banach space c_0, namely, the space of all real-valued sequences converging to 0. Let

$$A = \{x \in c_0 : x_n = 1 + 1/n \text{ for an odd number of } n\text{'s, and } x_n = 0 \text{ otherwise}\};$$

$$B = \{x \in c_0 : x_n = 1 + 1/n \text{ for an even number of } n\text{'s, and } x_n = 0 \text{ otherwise}\}.$$

One can see that $A, B \in \mathcal{H}(X)$ and $D(A,B) = 1$, but $N_{1/2}(A) \cap N_{1/2}(B) = \varnothing$, hence there is no geodesic connecting A to B. If X is assumed to be both geodesic and compact, then $\mathcal{H}(X)$ is geodesic as well [**10**], however, the assumption of compactness is somewhat restrictive.

THEOREM 2.1. *Let $X = (X, d)$ be a length space. Then $\mathcal{H}(X; C)$ is a length space for all $C \in \mathrm{CL}(X)$.*

PROOF. Choose a number $\sigma > 1$. For each $x, y \in X$, let $\gamma_{xy} : [0,1] \to X$ be a curve of length at most $\sigma d(x,y)$ such that $\gamma_{xy}(0) = x$ and $\gamma_{xy}(1) = y$. By our convention, γ_{xy} is parametrized proportional to its arclength, which implies $d(\gamma_{xy}(s), \gamma_{xy}(t)) \le \sigma|s-t|d(x,y)$ for all $0 \le s < t \le 1$.

Let $A, B \in \mathcal{H}(X; C)$ and let $E = \{(a,b) \in A \times B : d(a,b) \le \sigma D(A,B)\}$. By the definition of the Hausdorff metric,

$$\pi_A(E) = A \quad \text{and} \quad \pi_B(E) = B, \tag{2.1}$$

where π_A, π_B are the coordinate projections on $A \times B$.

Define a function Γ_{AB} from $[0,1]$ to the power set of X as follows:

$$\Gamma_{AB}(t) = \overline{\{\gamma_{ab}(t) : (a,b) \in E\}}.$$

Then $\Gamma_{AB}(t)$ is closed, and contained in the closed $\sigma^2 D(A,B)$-neighborhood of A. Thus $\Gamma_{AB} : [0,1] \to \mathcal{H}(X;C)$. By (2.1), $\Gamma_{AB}(0) = A$ and $\Gamma_{AB}(1) = B$. We claim that Γ_{AB} is a rectifiable curve of length at most $\sigma^2 D(A,B)$, although it need not be parametrized proportional to the arclength. Let s and t be given, $0 \le s < t \le 1$. For each $(a,b) \in E$

$$d(\gamma_{ab}(s), \gamma_{ab}(t)) \le \sigma|s-t|d(a,b) \le \sigma^2|s-t|D(A,B),$$

which implies

$$D(\Gamma_{AB}(s), \Gamma_{AB}(t)) \le \sigma^2|s-t|D(A,B). \tag{2.2}$$

Thus the length of Γ_{AB} does not exceed $\sigma^2 D(A,B)$. Since $\sigma > 1$ is arbitrary, $\mathcal{H}(X;C)$ is a length space. $\square$

COROLLARY 2.2. *Let X be a proper geodesic space. Then $\mathcal{H}(X;C)$ is a geodesic space for all $C \in \mathrm{CL}(X)$.*

PROOF. Since X is proper, for any $a \in X$ and any $B \in \mathrm{CL}(X)$ there exists $b \in B$ such that $d(a,b) = \mathrm{dist}(a,B)$. This and the fact that X is a geodesic space allow us to carry out the proof of Theorem 2.1 with $\sigma = 1$. Then (2.2) takes the form

$$D(\Gamma_{AB}(s), \Gamma_{AB}(t)) \le |s-t|D(A,B).$$

Since $\Gamma_{AB}(0) = A$ and $\Gamma_{AB}(1) = B$,

$$\begin{aligned} D(A,B) &\le D(\Gamma_{AB}(0), \Gamma_{AB}(s)) + D(\Gamma_{AB}(s), \Gamma_{AB}(t)) + D(\Gamma_{AB}(t), \Gamma_{AB}(1)) \\ &\le (s + |s-t| + 1 - t)D(A,B) \\ &= D(A,B). \end{aligned}$$

Thus equality holds throughout, and we conclude that

$$D(\Gamma_{AB}(s), \Gamma_{AB}(t)) = |s-t|D(A,B)$$

for all $A, B \in \mathcal{H}(X;C)$ and all $0 \le s < t \le 1$. $\square$

REMARK 2.3. Geodesics in $\mathcal{H}(X;C)$ are not unique if X is a geodesic space containing more than one point. Consider $\mathcal{H}([0,1])$ as an example. The sets $A = \{0\}$ and $B = [0,1]$ are at distance 1 from each other. The following two curves are geodesics connecting A to B:

$$\Gamma_1(t) = [0,t], \quad 0 \le t \le 1;$$
$$\Gamma_2(t) = \begin{cases} \{t\}, & 0 \le t \le 1/2; \\ [1-t,t], & 1/2 \le t \le 1. \end{cases}$$

If X is a geodesic space containing more than one point, then $\mathcal{H}(X;C)$ contains a rescaled copy of $\mathcal{H}([0,1])$ (see the proof of Proposition 2.5) and therefore has non-unique geodesics.

REMARK 2.4. In spite of Remark 2.3, it sometimes happens that two sets $A, B \in \mathcal{H}(X;C)$ can be connected by only one geodesic. Indeed, if $\Gamma : [0,1] \to \mathcal{H}(X;C)$ is any geodesic between A and B, then for every $t \in [0,1]$ the inequalities $D(\Gamma(t), A) \le tD$ and $D(\Gamma(t), B) \le (1-t)D$ imply

$$\Gamma(t) \subset N_{tD}(A) \cap N_{(1-t)D}(B),$$

where $D = D(A,B)$. If any proper subset E of $N_{tD}(A) \cap N_{(1-t)D}(B)$ satisfies either $D(E,A) > tD$ or $D(E,B) > (1-t)D$, then Γ is a unique geodesic connecting A to B. An example of this kind is given after Proposition 2.5.

It is interesting to determine which properties of metric spaces are inherited by their hyperspaces. The following proposition provides a negative result of this kind. A geodesic metric space X is *Gromov hyperbolic* if each side of a geodesic triangle is contained in the δ-neighborhood of the union of the other two sides. See, e.g., [**9**, Chapter III.H].

PROPOSITION 2.5. *Let X be a proper geodesic space. The hyperspace $\mathcal{H}(X;C)$ is Gromov hyperbolic if and only if X is bounded.*

PROOF. If X is bounded, then so is $\mathcal{H}(X;C)$, and bounded spaces are trivially Gromov hyperbolic.

Suppose that X is unbounded. Pick two points $a, b \in X$ at distance L from each other. Let $C' = C \setminus B(a, 3L)$ and let $\gamma_{ab} : [0,1] \to X$ be a geodesic connecting a to b. The Hausdorff distance between any two of the sets $C' \cup \{a\}$, $C' \cup \{b\}$ and $G = C' \cup \gamma_{ab}([0,1])$ is equal to L. For $0 \le t \le 1$ let $\Gamma_{\{a\}\{b\}}(t) = C' \cup \{\gamma_{ab}(t)\}$, $\Gamma_{\{a\}G}(t) = C' \cup \gamma_{ab}([0,L])$, and $\Gamma_{G\{b\}}(t) = C' \cup \gamma_{ab}([L,1])$. These are geodesics in $\mathcal{H}(X;C)$. For every t the sets $\Gamma_{\{a\}G}(t)$ and $\Gamma_{G\{b\}}(t)$ contain at least one of the points a and b. It follows that the distance in $\mathcal{H}(X;C)$ from $\Gamma_{\{a\}\{b\}}(1/2)$ to $\Gamma_{\{a\}G}([0,1]) \cup \Gamma_{G\{b\}}([0,1])$ is equal to $L/2$. Since L can be arbitrarily large, $\mathcal{H}(X;C)$ is not Gromov hyperbolic. $\square$

Despite Proposition 2.5, one can sometimes find large (unbounded) subsets of $\mathcal{H}(X)$ that are Gromov hyperbolic. To this end, we introduce *uniformly bounded hyperspaces*

$$\mathcal{H}_L(X) = \{A \in \mathcal{H}(X) : \operatorname{diam} A \le L\}.$$

Unfortunately, the analogue of Corollary 2.2 is false for $\mathcal{H}_L(X)$. Indeed, let $S(R) = \{x \in \mathbb{R}^n : |x| = R\}$ be a sphere equipped with the intrinsic metric. Choose $h \in (0,R)$ and let $A = \{x \in S(R) : x_n = h\}$, $B = \{x \in S(R) : x_n = -h\}$.

Clearly $D(A,B) = 2R\sin^{-1}(h/R)$. Using Remark 2.4 we find that there is only one geodesic connecting A to B in $\mathcal{H}(X)$, namely

$$\Gamma_{AB}(t) = \{x \in S(R) : x_n = R\sin(\sin^{-1}(h/R)(1-2t))\}.$$

Since $\operatorname{diam}\Gamma_{AB}(1/2) = \pi R$, it follows that $\mathcal{H}_L(S(R))$ is not geodesic for any $L < \pi R$.

The preceding example is bounded, hence trivially Gromov hyperbolic, but fails an additional test of nonpositive curvature. A geodesic metric space X is *nonpositively Busemann curved* if for any two geodesics $\gamma : [a,b] \to X$ and $\gamma' : [a',b'] \to X$, the map $(t,t') \mapsto d(\gamma(t),\gamma'(t))$ from $[a,b]\times[a',b']$ to $[0,\infty)$ is convex. See, e.g., [**27**, Chapter 8] or [**13**].

Proposition 2.6. *If X is a proper nonpositively Busemann curved space, then $\mathcal{H}_L(X)$ is a geodesic space for all $L \geq 0$.*

Proof. Given $A,B \in \mathcal{H}_L(X)$, let Γ_{AB} be as in the proof of Theorem 2.1 (with $\sigma = 1$). Let $\gamma = \gamma_{ab}$ and $\gamma' = \gamma_{a'b'}$, where $a,a' \in A$ and $b,b' \in B$. The convexity of the distance function in nonpositively curved Busemann spaces (see, e.g., [**27**, Proposition 8.1.2]) yields

$$d(\gamma(t),\gamma'(t)) \leq (1-t)d(\gamma(0),\gamma'(0)) + td(\gamma(1),\gamma'(1)) \leq (1-t)L + tL = L.$$

Thus $\operatorname{diam}\Gamma_{AB}(t) \leq L$ as required. □

Examples of nonpositively curved Busemann spaces include CAT(0) spaces [**9**, p. 176] and strictly convex normed vector spaces [**27**, Proposition 8.1.6].

Under the assumptions of Proposition 2.6 the space $\mathcal{H}_L(X)$ does not have to be a nonpositively curved Busemann space itself. This follows from Remark 2.3 and the uniqueness of geodesics in such spaces [**27**, Proposition 8.1.4].

Proposition 2.7. *Let X be a proper nonpositively curved Busemann space. If X is δ-hyperbolic, then $\mathcal{H}_L(X)$ is δ'-hyperbolic, where δ' depends on δ and L. Furthermore, the Gromov boundaries of X and $\mathcal{H}_L(X)$, equipped with visual metrics with common visual parameter, are bi-Lipschitz equivalent.*

We refer to [**9**, Chapter III.H] for the definition and basic properties of the Gromov boundary and visual metrics thereon.

Proof. Let $s : \mathcal{H}_L(X) \to X$ be a mapping such that $s(A) \in A$ for every $A \in \mathcal{H}_L(X)$ (the existence of such s follows from the Axiom of Choice). Let A and B be two sets in $\mathcal{H}_L(X)$, and let $D = D(A,B)$, $d = d(s(A),s(B))$. Since $s(A) \in N_D(B) \subset N_{D+L}(\{s(B)\})$, we have $d \leq D + L$. Conversely, $A \subset N_L(\{s(A)\}) \subset N_{d+L}(B)$ and $B \subset N_{d+L}(A)$, which means that $d \geq D - L$. Thus the mapping s is a $(1,L)$-quasi-isometry. Since $\mathcal{H}_L(X)$ admits a quasi-isometric embedding into a δ-hyperbolic space, it is δ'-hyperbolic with $\delta' = \delta'(L,\delta)$ [**9**, p. 402]. By a simple modification of the argument in the proof of Theorem 6.5 (1) of [**7**], the mapping s induces a bi-Lipschitz mapping $\partial s : \partial\mathcal{H}_L(X) \to \partial X$. Finally, since s is surjective, so is ∂s (Proposition 6.3 (4) of [**7**]). □

We now turn to the hyperspaces of convex sets: $\mathcal{H}^c(X)$ and $\mathcal{H}^c(X;C)$. Corollary 2.2 fails for $\mathcal{H}^c(X)$, which can be seen as follows. As before, let $S(R) = \{x \in \mathbb{R}^n : |x| = R\}$ be a sphere equipped with the intrinsic metric. Choose $h \in (0,R)$ and let $A = \{x \in S(R) : x_n \geq h\}$, $B = \{x \in S(R) : x_n = -R\}$. Clearly $A,B \in \mathcal{H}^c(S(R))$ and $D(A,B) = \pi R$. Suppose that $\gamma : [0,1] \to \mathcal{H}^c(S(R))$ is a geodesic segment joining A to B. Choose three points $a, b, c \in A$ so that the geodesic triangle $\triangle(a, b, c)$ in $S(R)$ with vertices a, b and c contains the point $p = (0, 0, \dots, R)$ in its interior. For all sufficiently small $t > 0$ the set $\gamma(t)$ must contain points a', b' and c' such that the interior of the geodesic triangle $\triangle(a', b', c')$ also contains p. Since A_t is geodesically convex, we have $P \in A_t$, hence $D(A_t, B) = \pi R = D(A, B)$. This is a contradiction.

Once again, nonpositive curvature bounds save the day. When X is nonpositively curved in the sense of Busemann, geodesics in X are unique, which allows us to define the convexification map $\mathrm{conv} : \mathcal{H}(X; C) \to \mathrm{CL}(X)$ as follows: $\mathrm{conv}(A)$ is the intersection of all geodesically convex closed sets containing A. To prove that the hyperspaces of convex subsets of nonpositively curved Busemann spaces are geodesic, we require the following lemma.

LEMMA 2.8. *If X is a proper nonpositively curved Busemann space, then the convexification map is a contraction from $\mathcal{H}(X; C)$ into $\mathcal{H}^c(X; \mathrm{conv}(C))$.*

The version of Lemma 2.8 for $\mathcal{H}(X)$ was proved in [**11**] (Lemma 3.1). Our proof is essentially the same and is given here for the reader's convenience.

PROOF. Given $A \in \mathcal{H}(X; C)$, one can write $\mathrm{conv}(A)$ as the closure of an increasing union of sets A_m, where $A_0 = A$ and A_{m+1} is the union of all geodesics joining points of A_m. To prove that conv is a contraction, it suffices to show $D(\overline{A}_{m+1}, \overline{B}_{m+1}) \le D(\overline{A}_m, \overline{B}_m)$ for all m, or equivalently, just for $m = 0$. For every $a \in A_1$ there exist a geodesic segment $\gamma : [0, 1] \to X$ such that $\gamma(0), \gamma(1) \in A$ and $\gamma(t) = a$ for some t. Choose another geodesic $\gamma' : [0, 1] \to X$ so that $\gamma'(0), \gamma'(1) \in B$, $d(\gamma(0), \gamma'(0)) \le D(A, B)$, and $d(\gamma(1), \gamma'(1)) \le D(A, B)$. Using the convexity of the distance function, we obtain

$$d(a, \gamma'(t)) \le t d(\gamma(0), \gamma'(0)) + (1 - t) d(\gamma(1), \gamma'(1)) \le D(A, B).$$

Since $\gamma'(t) \in B_1$, it follows that $\overline{A}_1 \subset N_{D(A,B)}(\overline{B}_1)$. Interchanging the roles of A and B, we obtain $D(\overline{A}_1, \overline{B}_1) \le D(A, B)$, as required. $\square$

When $\mathrm{conv}(C) \notin \mathcal{H}(X; C)$, the convexification map is no longer a retraction of $\mathcal{H}(X; C)$. Moreover, $\mathrm{conv} : \mathcal{H}(X; C) \to \mathcal{H}^c(X; \mathrm{conv}(C))$ need not be a surjection. Indeed, let $X = \mathbb{R}^2$ and $C = \{(n^2, 0) : n = 1, 2, \dots\}$. Given $A \in \mathcal{H}(X; C)$, let A' be the projection of A onto the x-axis. Let $d = D(A', C)$; clearly $d \le D(A, C) < \infty$. Fix an integer $n \ge d$. The definition of d implies

$$\begin{aligned} &A' \cap [n^2 - d, n^2 + d] \neq \varnothing; \\ &A' \cap (n^2 + d, (n + 1)^2 - d) = \varnothing; \\ &A' \cap [(n + 1)^2 - d, (n + 1)^2 + d] \neq \varnothing. \end{aligned}$$

In other words, the vertical strip $S = \{(x, y) : n^2 + d < x < (n + 1)^2 - d\}$ separates A. It follows that $\mathrm{conv}(A) \cap S$ is a (possibly degenerate) quadrangle. In particular, $\mathrm{conv}(A)$ is not strictly convex. On the other hand, $\mathcal{H}^c(X; \mathrm{conv}(C))$ contains some strictly convex sets, e.g., $\{(x, y) : x \ge 0, |y| \le \tan^{-1} x\}$. Therefore, $\mathrm{conv}(\mathcal{H}(X; C)) \neq \mathcal{H}^c(X; \mathrm{conv}(C))$.

COROLLARY 2.9. *If X is a proper nonpositively curved Busemann space, then $\mathcal{H}^c(X; C)$ is a geodesic space for all convex sets $C \in \mathrm{CL}(X)$.*

PROOF. By virtue of Corollary 2.2 and Lemma 2.8 the proof reduces to the following simple observation. If γ is a geodesic in a metric space Y, and $T : Y \to Y$ is a contraction that fixes the endpoints of γ, then $T \circ \gamma$ is a geodesic in $T(Y)$. $\square$

If C is bounded, Corollary 2.9 reduces to Proposition 3.5 in [**11**].

3. Mappings of hyperspaces

Every continuous mapping $f : X \to Y$ induces a mapping $G_f : \mathrm{CL}(Y) \to \mathrm{CL}(X)$ as follows: $G_f(A) = f^{-1}(A)$. In general G_f is not continuous, and can even map two sets at finite distance from each other into sets at infinite distance. Indeed, let $X = \{(x, y) \in \mathbb{R}^2 : x > 0\} \cup \{(x, 0) \in \mathbb{R}^2 : x \le 0\}$, $Y = \mathbb{R}$, and define $f : X \to Y$ by $f(x, y) = x$. Obviously $D(G_f(\{0\}), G_f(\{1\})) = \infty$. Observe that f is an open mapping, i.e. for any $x \in X$ and $r > 0$ there is $\rho > 0$ such that $B(f(x), \rho) \subset f(B(x, r))$. However, ρ cannot be chosen independently of r. This leads us to the following definition, which appeared, e.g. in [**20**].

DEFINITION 3.1. A mapping $f : X \to Y$ between two metric spaces X, Y is called *co-uniformly continuous* if there exists an increasing function $\tilde{\omega}_f : (0, \infty) \to (0, \infty)$ such that $B(f(x), \tilde{\omega}_f(r)) \subset f(B(x, r))$ for all $x \in X$ and all $r > 0$.

If $f : X \to Y$ is co-uniformly continuous, then G_f is uniformly continuous. Indeed, given two sets $A, B \in \mathrm{CL}(Y)$ such that $D(A, B) \le \tilde{\omega}_f(r)$, we easily obtain $f^{-1}(A) \subset N_r(f^{-1}(B))$ and $f^{-1}(B) \subset N_r(f^{-1}(A))$. Similarly, if f is a Lipschitz quotient (defined in the introduction), then G_f is bi-Lipschitz (cf. Lemma 6.1 [**25**]).

Every uniformly continuous homeomorphism $f : X \to Y$ between metric spaces lifts to a continuous mapping $F_f : \mathcal{H}(X) \to \mathcal{H}(Y)$ of the corresponding hyperspaces:

$$F_f(A) = \{f(a) : a \in A\}.$$

If f is bi-Lipschitz, then F_f is also bi-Lipschitz. In other words, the class of bi-Lipschitz mappings is invariant under the hyperspace functor. The same is true for isometries, and moreover, for many spaces X every isometry of $\mathcal{H}(X)$ coincides with F_f where f is an isometry of X [**3, 13**]. The following result shows that quasisymmetric mappings are not preserved by the hyperspace functor. Before stating it, let us introduce the pointwise Lipschitz constant of f at $x \in X$ by

$$L_f(x) := \limsup_{y \to x} \frac{d(f(x), f(y))}{d(x, y)}.$$

We say that a map $f : X \to Y$ is *hyperquasisymmetric* if F_f is quasisymmetric. Considering the action of F_f on singletons, we see that such f must be quasisymmetric itself.

PROPOSITION 3.2. *Let $f : X \to Y$ be a homeomorphism of connected metric spaces which is Lipschitz on a neighborhood of one point in X, and whose pointwise Lipschitz constant is infinite at another point in X. Then f is not hyperquasisymmetric.*

For example, the map $f : [0, 1] \to [0, 1]$, $f(t) = \sqrt{t}$, is quasisymmetric, but not hyperquasisymmetric.

PROOF. We use primes to denote images under f and F_f, i.e., $u' = f(u)$, $A' = F_f(A)$, etc. Let U be a neighborhood of a point $p \in X$ such that $f|_U$ is

L-Lipschitz, and let $q \in X \setminus U$ satisfy $L_f(q) = +\infty$. We may assume without loss of generality that $U = B(p, \epsilon)$ for some

$$0 < \epsilon < \min\left\{\frac{1}{3}d(p,q), \frac{1}{2L+1}d(p',q')\right\}.$$

Choose points $v_n \to q$ so that $d(v_n', q') > nd(v_n, q)$. By restricting our attention to sufficiently large n, we may assume that

$$\max\{d(v_n,q), d(v_n',q')\} \le \epsilon.$$

Then

$$\min\{d(v_n,p), d(v_n',p')\} \ge \epsilon$$

for all n.

Since X is connected, we may choose points $u_n \in U$ so that $\delta_n := d(p,u_n) = d(q,v_n)$ for all n. Let $A_n = \{p, v_n\}$, $B_n = \{p, u_n, v_n\}$, and $C_n = \{p, v_n, q\}$. Then $D(A_n, B_n) = D(A_n, C_n) = \delta_n$. On the other hand,

$$D(A_n', B_n') = \min\{d(p', u_n'), d(u_n', v_n')\} \le L\delta_n$$

while

$$D(A_n', C_n') = \min\{d(p', q'), d(v_n', q')\} > n\delta_n.$$

Thus F_f is not quasisymmetric. $\square$

The main theorem of this section provides two characterizations of the class of hyperquasisymmetric maps and incidentally shows that this class is invariant under the hyperspace functor. In other words, hyperquasisymmetric maps lift to quasisymmetric maps on iterated hyperspaces such as $\mathcal{H}(\mathcal{H}(X))$. See [**2, 31, 34**] for more on iterated hyperspaces. Some preliminaries are required to state the theorem. Given a metric space X, let $\mathrm{Dist}(X)$ be its distance set:

$$\mathrm{Dist}(X) = \{d_X(x_1, x_2) : x_1, x_2 \in X\} \subset [0, \infty).$$

The modulus of continuity of a mapping $f : X \to Y$ is defined as

$$\omega_f(\delta) = \sup\{d_Y(f(x_1), f(x_2)) : d_X(x_1, x_2) \le \delta\}, \quad \delta > 0.$$

We say that ω_f is controlled by a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ provided that $\omega_f(\delta) < \infty$ for all $\delta > 0$ and

$$\omega_f(t\delta) \le \eta(t)\omega_f(\delta), \quad \delta \in \mathrm{Dist}(X),\ t > 0.$$

Whether or not ω_f is controlled by η may depend on X as well as on ω_f and η.

LEMMA 3.3. *Let $C \in \mathrm{CL}(X)$. For any $\delta \in \mathrm{Dist}(\mathcal{H}(X; C))$ there exists a non-decreasing sequence $\{\delta_n\} \subset \mathrm{Dist}(X)$ such that $\delta_n \to \delta$.*

PROOF. Choose $A, B \in \mathcal{H}(X; C)$ so that $D(A, B) = \delta$. Without loss of generality we may assume that for any $\epsilon > 0$ there is $a \in A$ such that $\mathrm{dist}(a, B) \ge \delta - \epsilon$. Then there is $b \in B$ such that $\delta - \epsilon \le d_X(a, b) \le \delta$. Since $\epsilon > 0$ is arbitrary, the claim follows. $\square$

THEOREM 3.4. *Let f be a homeomorphism of a metric space X into a metric space Y. The following are equivalent:*

(i) f is hyperquasisymmetric;

(ii) there exists a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that for any four distinct points $x_1, \dots, x_4 \in X$

$$\frac{d_Y(f(x_1), f(x_2))}{d_Y(f(x_3), f(x_4))} \le \eta\left(\frac{d_X(x_1,x_2)}{d_X(x_3,x_4)}\right); \tag{3.1}$$

(iii) ω_f is controlled by some homeomorphism $\eta : [0,\infty) \to [0,\infty)$ and there exists $c > 0$ such that for all $x_1, x_2 \in X$

$$d_Y(f(x_1), f(x_2)) \ge c\,\omega_f(d_X(x_1,x_2)); \tag{3.2}$$

(iv) F_f is hyperquasisymmetric.

Remark 3.5. The mappings satisfying (3.1) have previously appeared in the literature on several occasions. Ivascu [**19**] called them "freely quasisymmetric"; we do not use this term to avoid confusion with Väisälä's well-known terminology [**35**]. Aseev and Shalaginov [**1**] studied such mappings in the context of self-similar spaces. They proved, among other things, that a mapping $f : X \to Y$ satisfies (3.1) if and only if it lifts to a quasisymmetric mapping $f \times f : X \times X \to Y \times Y$. See also [**30**]. Ghamsari and Herron [**14**] and Herron and Mayer [**17**] studied this class of maps in connection with the question of characterizing bi-Lipschitz homogeneous Jordan curves. Combining Theorem 3.4 with Theorem E of [**17**] leads to the following result: a Jordan curve Γ in a doubling metric space is bi-Lipschitz homogeneous and bounded turning if and only if it admits a hyperquasisymmetric parametrization.

Proof of Theorem 3.4. (i)$\Rightarrow$(ii). Suppose that F_f is η-quasisymmetric. Choose distinct points $x_1, \dots, x_4 \in X$ and set $A = \{x_1, x_3\}$, $B = \{x_2, x_3\}$ and $C = \{x_1, x_4\}$. Then

$$D(A,C) = \min\{d_X(x_3,x_4), \max\{d_X(x_1,x_3), d_X(x_1,x_4)\}\} \ge \frac{1}{2} d_X(x_3,x_4)$$

and

$$D(A,B) = \min\{d_X(x_1,x_2), \max\{d_X(x_1,x_3), d_X(x_2,x_3)\}\} \le d_X(x_1,x_2).$$

Similar considerations show that

$$D(F_f(A), F_f(B)) \ge \frac{1}{2} d_Y(f(x_1), f(x_2))$$

and

$$D(F_f(A), F_f(C)) \le d_Y(f(x_3), f(x_4)).$$

Using the quasisymmetry of F_f, we find

$$\frac{d_Y(f(x_1), f(x_2))}{d_Y(f(x_3), f(x_4))} \le 2\eta\left(2\frac{d_X(x_1,x_2)}{d_X(x_3,x_4)}\right).$$

Thus f satisfies (ii) with the function $\eta_1(t) = 2\eta(2t)$.

(ii)$\Rightarrow$(iii). The first part of (iii) follows immediately from (ii). To prove (3.2), let $\delta = d_X(x_1,x_2)$, pick $\epsilon > 0$ and choose $x_3, x_4 \in X$ so that $d_X(x_3,x_4) \le \delta$ and $d_Y(f(x_3), f(x_4)) \ge \omega_f(\delta) - \epsilon$. Using (3.1) with x_i appropriately rearranged, we obtain

$$d_Y(f(x_1), f(x_2)) \ge \eta(1)^{-1}(\omega_f(\delta) - \epsilon),$$

and (iii) follows with $c = 1/\eta(1)$.

(iii)⇒(i). First we prove that F_f itself has the properties listed in (iii). It is easy to see that F_f has the same modulus of continuity as f. Given $\delta \in \mathrm{Dist}(\mathcal{H}(X))$, let $\{\delta_n\}$ be as in Lemma 3.3. For any $t > 0$ we have

$$\omega_f(t\delta) \le \eta(t\delta_n/\delta)\omega_f(\delta_n) \le \eta(t\delta_n/\delta)\omega_f(\delta).$$

Letting $n \to \infty$ yields $\omega_f(t\delta) \le \eta(t)\omega_f(\delta)$. Thus the modulus of continuity of F_f is controlled by η.

To prove that F_f satisfies (3.2), pick $A, B \in \mathcal{H}(X)$ and let $s = D(A, B)$. We may assume that there exists a sequence $\{a_n\} \subset A$ such that $s_n := \mathrm{dist}_X(a_n, B) \to s$; otherwise interchange A and B. Passing to a subsequence, we can ensure that one of the following two cases occurs.

Case 1. For each n there is $b_n \in B$ such that $d_X(a_n, b_n) = s_n$. Inequality (3.2) implies

$$\mathrm{dist}_Y(f(a_n), f(B)) \ge c\omega_f(s_n). \tag{3.3}$$

Since $s_n \in \mathrm{Dist}(X)$, it follows that

$$\lim_{n\to\infty} \omega_f(s_n) \ge \lim_{n\to\infty} \eta(s/s_n)^{-1}\omega_f(s) = \eta(1)^{-1}\omega_f(s). \tag{3.4}$$

Combining (3.3) and (3.4) we obtain $D(f(A), f(B)) \ge c\eta(1)^{-1}\omega_f(s)$. Thus F_f satisfies (3.2) with the constant $c' = c/\eta(1)$.

Case 2. the distance from a_n to B is not achieved for any n. Inequality (3.2) implies

$$\mathrm{dist}_Y(f(a_n), f(B)) \ge c \lim_{\epsilon\downarrow 0} \omega_f(s_n + \epsilon). \tag{3.5}$$

Choose a sequence $\{b_{mn}\} \subset B$ such that $d_X(a_n, b_{mn}) \downarrow s_n$ as $m \to \infty$. Since $d_X(a_n, b_{mn}) \in \mathrm{Dist}(X)$, we have

$$\omega_f(s) \le \eta(s/s_n) \lim_{\epsilon\downarrow 0} \omega_f(s_n + \epsilon).$$

The latter inequality and (3.5) yield $D(f(A), f(B)) \ge c\eta(1)^{-1}\omega_f(s)$. This completes the proof of property (3.2) for F_f.

Finally, for any distinct $A, B, C \in \mathcal{H}(X)$ we have

$$\frac{D(f(A), f(B))}{D(f(A), f(C))} \le \frac{\omega_f(D(A,B))}{c'\omega_f(D(A,C))} \le \frac{1}{c'}\eta\left(\frac{D(A,B)}{D(A,C)}\right),$$

which proves (i).

(i)⇒(iv). If f is hyperquasisymmetric, then it satisfies (iii), and so does F_f (by the previous step). Since (iii) implies (i), F_f is hyperquasisymmetric. □

Remark 3.6. The proof of implication (iii)⇒(i) works for unbounded closed sets as well. Therefore, a hyperquasisymmetric mapping $f : X \to Y$ induces (hyper-)quasisymmetric mappings on all components of $\mathrm{CL}(X)$.

Corollary 3.7. *Let X be a Gromov hyperbolic space, and let d' and d'' be two visual metrics on its boundary ∂X. Then the identity map $\mathrm{id} : (\partial X, d') \to (\partial X, d'')$ is hyperquasisymmetric.*

Proof. There exist $\alpha > 0$ and $C > 1$ such that

$$C^{-1}(d'(p,q))^\alpha \le d''(p,q) \le C(d'(p,q))^\alpha$$

for all $p, q \in \partial X$ (see, e.g. Proposition III.H.3.21 in [**9**]). This and Theorem 3.4 imply that id is hyperquasisymmetric. □

Corollary 3.7 can be generalized as follows. Bonk and Schramm [**7**] call $f : X \to Y$ a *snowflake map* if there exist $\alpha > 0$ and $C \geq 1$ such that

$$C^{-1} d_X(x_1, x_2)^\alpha \leq d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2)^\alpha$$

for all $x_1, x_2 \in X$. By Theorem 3.4 snowflake maps are hyperquasisymmetric. Examples of snowflake maps are provided by Theorem 6.5 [**7**], which asserts that rough similarities between Gromov hyperbolic spaces induce snowflake maps on their boundaries.

4. Hyperconformal dimension

The following definition is motivated by Corollary 3.7.

DEFINITION 4.1. Let X be a metric space. The *hyperconformal* dimension of X is defined by

$$\operatorname{HCdim} X = \inf\{\operatorname{Hdim} f(X) : f \text{ is hyperquasisymmetric}\},$$

where Hdim stands for the Hausdorff dimension [**24**].

Definition 4.1 is modeled after the more commonly used definition of *conformal dimension* [**6, 16, 26**]:

$$\operatorname{Cdim} X = \inf\{\operatorname{Hdim} f(X) : f \text{ is quasisymmetric}\}.$$

Evidently $\operatorname{Cdim} X \leq \operatorname{HCdim} X \leq \operatorname{Hdim} X$. By virtue of Corollary 3.7 both conformal and hyperconformal dimensions of the boundary of a hyperbolic space are well-defined. Moreover, conformal dimension of ∂X is invariant under quasi-isometries of X, while hyperconformal dimension is invariant under rough similarities of X [**7**].

Since hyperquasisymmetric maps of metric spaces preserve more structure than general quasisymmetric maps, one can expect $\operatorname{HCdim} X$ to reveal some features of X that are not recorded by $\operatorname{Cdim} X$. For instance, conformal dimension does not distinguish between spaces X with $\operatorname{Cdim} X < 1$ [**22, 33**]. This is in contrast with the following result.

THEOREM 4.2. *For every $s \geq 0$ there exists a compact metric space X with $\operatorname{HCdim} X = s$.*

PROOF. For each $s \in \{0\} \cup [1, \infty)$ there exists a compact metric space X such that $\operatorname{Cdim} X = \operatorname{Hdim} X = s$ and therefore $\operatorname{HCdim} X = s$ [**5, 8, 26, 32**]. It remains to consider the case $0 < s < 1$. For any such s we shall construct $X \subset \mathbb{R}$ as a union of two generalized Cantor sets [**24**, 4.11]. Given an infinite set of positive integers $E \subset \{1, 2, \dots\}$, define a sequence $\lambda^E = \{\lambda_n^E : n \geq 1\}$ so that $\lambda_n^E = 1/2$ for $n \notin E$ and $\prod_{k=1}^n \lambda_k^E = 2^{-n/s}$ for $n \in E$. Let

$$C(\lambda^E) = \bigcap_{n=0}^{\infty} \bigcup_{j=1}^{2^n} I_j^n,$$

where $I_1^0 = [0, 1]$ and the other closed intervals I_j^n are defined as follows. The intervals $\{I_j^n\}_{j=1}^{2^n}$ are disjoint, each of length equal to $\mu_n^E := \prod_{k=1}^n \lambda_k^E$; furthermore, each I_j^n is contained in an interval I_i^{n-1} and shares an endpoint with it.

By virtue of [**24**, 4.11] we have $\operatorname{Hdim} C(\lambda^E) = s$ for any E as above. Let n_i be a strictly increasing sequence of integers such that

$$n_{i+2} - n_i > \frac{n_{i+1} - n_i}{s} + 1, \quad i \geq 1, \tag{4.1}$$

and

$$\lim_{i\to\infty} \frac{n_i}{i} = \infty. \tag{4.2}$$

Define $E = \{n_{2i-1} : i \geq 1\}$, $F = \{n_{2i} : i \geq 1\}$, and $X = C(\lambda^E) \cup (1 + C(\lambda^F)) \subset \mathbb{R}$. Clearly $\operatorname{Hdim} X = s$. We claim that $\operatorname{HCdim} X = s$, or, equivalently, $\operatorname{Hdim} f(X) \geq s$ for any hyperquasisymmetric mapping f.

A generalized Cantor set such as $C(\lambda^E)$ contains the endpoints of all intervals I_j^n used in its construction. Therefore, for any $i \geq 1$ the set X contains a pair of points a_i, b_i such that $b_i - a_i = 2^{-n_i/s}$. Furthermore, X contains the points that split $[a_i, b_i]$ into $2^{n_{i+2}-n_i-1}$ equal subintervals. Suppose that $f : X \to Y$ is hyperquasisymmetric, and $c > 0$ is as in (3.2). For every integer j between 1 and $n_{i+2} - n_i - 1$ we have

$$\omega_f(2^{-j-n_i/s}) \geq 2^{-j} d_Y(f(a_i), f(b_i)) \geq 2^{-j} c\omega_f(2^{-n_i/s}). \tag{4.3}$$

Setting j equal to the smallest integer greater than $(n_{i+1} - n_i)/s$ (cf. (4.1)), we obtain

$$\omega_f(2^{-n_{i+1}/s}) \geq 2^{-(n_{i+1}-n_i)/s-1} c\omega_f(2^{-n_i/s}).$$

Therefore, for all $i > 1$

$$\omega_f(2^{-n_i/s}) \geq \left(\frac{c}{2}\right)^{i-1} 2^{-(n_i-n_1)/s} \omega_f(2^{-n_1/s}) =: C_1 \left(\frac{c}{2}\right)^{i-1} 2^{-n_i/s}. \tag{4.4}$$

For any $\epsilon > 0$ inequality (4.4) implies

$$\lim_{i\to\infty} 2^{(1+\epsilon)n_i/s} \omega_f(2^{-n_i/s}) \geq C_1 \lim_{i\to\infty} 2^{\epsilon n_i/s} \left(\frac{c}{2}\right)^{i-1} = \infty,$$

where in the last step we used (4.2). This and (4.3) yield

$$\lim_{\delta\downarrow 0} \omega_f(\delta)\delta^{-1-\epsilon} = \infty.$$

Using (3.2) again, we conclude that

$$\operatorname{Hdim} f(X) \geq (1+\epsilon)^{-1} \operatorname{Hdim} X.$$

Since ϵ was arbitrary, $\operatorname{Hdim} f(X) \geq \operatorname{Hdim} X$ as desired. □

A metric space X is called Ahlfors regular if there exist $s > 0$ and $C \geq 1$ such that the s-dimensional Hausdorff measure H^s satisfies

$$C^{-1} r^s \leq H^s(B(x, r)) \leq C r^s$$

for all $x \in X$ and $0 < r \leq 2 \operatorname{diam} X$. Since the boundary of a Gromov hyperbolic group is Ahlfors regular, it is natural to ask the following

QUESTION 4.3. Are there any Ahlfors regular metric spaces with hyperconformal dimension between 0 and 1?

The following proposition shows that HCdim is much more rigid than Cdim.

PROPOSITION 4.4. *If X contains a nontrivial rectifiable curve, then* $\operatorname{HCdim} X = \operatorname{Hdim} X$.

PROOF. Let $\gamma : [0, L] \to X$ be a curve parametrized by arclength and such that $\gamma(0) \neq \gamma(1)$. Suppose that $f : X \to Y$ is surjective and hyperquasisymmetric. Our goal is to prove that $\operatorname{Hdim} Y \geq \operatorname{Hdim} X$. Given $n \geq 1$, let $x_i = \gamma(Li/n)$, $i = 0, \dots, n$. Since

$$\sum_{i=1}^{n} d_Y(f(x_i), f(x_{i-1})) \geq d_Y(f(\gamma(0)), f(\gamma(1))) =: s$$

and $d_X(x_i, x_{i-1}) \leq L/n$ for all i, it follows that $\omega_f(L/n) \geq s/n$. Using the fact that ω_f is nondecreasing, we obtain $\liminf_{\delta \to 0} \delta^{-1}\omega_f(\delta) > 0$. By property (iii) in Theorem 3.4, f^{-1} is Lipschitz on small scales. Thus $\operatorname{Hdim} Y \geq \operatorname{Hdim} X$. $\square$

Proposition 4.4 implies that the hyperconformal dimension of the Sierpiński carpet S is equal to $\operatorname{Hdim} S = \log 8/\log 3$. Although the precise value of $\operatorname{Cdim} S$ remains unknown, it is known to be strictly less than $\operatorname{Hdim} S$ [**6, 21**].

Bonk and Kleiner [**6**] introduced another version of Cdim, called *Ahlfors regular conformal dimension.* It is defined only for Ahlfors regular metric spaces X:

$$\operatorname{ARCdim} X = \inf\{\operatorname{Hdim} f(X) : f \text{ is quasisymmetric and } f(X) \text{ is Ahlfors regular}\}.$$

Clearly $\operatorname{Cdim} X \leq \operatorname{ARCdim} X \leq \operatorname{Hdim} X$. It is therefore of interest to compare ARCdim with HCdim. The Sierpiński carpet S has $\operatorname{ARCdim} S < \operatorname{HCdim} S$ [**21**]. It remains unclear if there is an Ahlfors regular space X with $\operatorname{ARCdim} X > \operatorname{HCdim} X$. No such spaces would exist if hyperquasisymmetric mappings preserved Ahlfors regularity; however, this is not the case.

PROPOSITION 4.5. *There exists a hyperquasisymmetric mapping of the real line $\mathbb{R}$ onto a metric space Y that is not Ahlfors regular.*

PROOF. Let Y be the real line $\mathbb{R}$ equipped with metric $d_Y(a, b) = \rho(|a - b|)$, where $\rho : [0, \infty) \to [0, \infty)$ is an increasing concave function to be determined. The identity map $\mathrm{id} : \mathbb{R} \to Y$ is η-quasisymmetric provided that

$$\frac{\rho(tu)}{\rho(u)} \leq \eta(t), \quad t, u > 0. \tag{4.5}$$

We shall choose ρ so that it is C^1 smooth on $(0, \infty)$ and

$$C^{-1} \leq \frac{u\rho'(u)}{\rho(u)} \leq C, \quad u > 0 \tag{4.6}$$

for some $C \geq 1$. It is easy to see that (4.6) yields (4.5) with $\eta(t) = \max\{t^C, t^{1/C}\}$. Theorem 3.4 implies that $\mathrm{id} : \mathbb{R} \to Y$ is hyperquasisymmetric as long as it is quasisymmetric. It remains to construct ρ so that (4.6) holds but Y is not Ahlfors regular. One possible choice is

$$\rho(u) = \begin{cases} u \log(3/u), & 0 < u \leq 1; \\ (\log 3 - 1)u + 1, & u > 1. \end{cases}$$

Indeed,

$$\rho'(u) = \begin{cases} \log(3/u) - 1, & 0 < u \leq 1; \\ \log 3 - 1, & u > 1, \end{cases}$$

hence ρ is concave and

$$1 - \frac{1}{\log 3} \leq \frac{u\rho'(u)}{\rho(u)} \leq 1, \quad u > 0.$$

It remains to observe that $\operatorname{Hdim} Y = 1$ but the 1-dimensional Hausdorff measure on Y is not locally finite. $\square$

5. Embeddings into hyperspaces

In this section we consider isometric and bi-Lipschitz embeddings of metric spaces into hyperspaces. Of course, every space X embeds isometrically into $\mathcal{H}(X)$ via the map $x \mapsto \{x\}$; our goal is to embed X into the hyperspace of a simpler metric space, such as $\mathbb{R}$. Before stating our first result, we introduce the notation $l^\infty_+ = \{x \in l^\infty : x_k \geq 0 \ \forall k\}$.

THEOREM 5.1. *Let X be a connected metric space. If $C \in \mathrm{CL}(X)$ is unbounded, then $\mathcal{H}(X; C)$ contains an isometric copy of l^∞_+. Furthermore, every separable metric space admits a $\sqrt{2}$-bi-Lipschitz embedding into $\mathcal{H}(X; C)$, and every bounded separable metric space admits an isometric embedding. The constant $\sqrt{2}$ is sharp.*

PROOF. Given $x = (x_1, x_2, \dots) \in l^\infty_+$, let $x_{ij} = \min\{x_i, j\}$ for $i, j = 1, 2, \dots$. The sequence $\tilde{x} := (x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}, \dots)$ satisfies $0 \leq \tilde{x}_k \leq k$ for all $k \geq 1$. Also, the map $x \mapsto \tilde{x}$ is an isometry of l^∞_+ into itself.

Since C is unbounded, it contains a sequence $\{c_n\}$ such that

$$d(c_n, c_0) \geq d(c_{n-1}, c_0) + 2n, \quad n \geq 1.$$

For $x \in l^\infty_+$ let

$$A(x) = \left(C \setminus \bigcup_{n=1}^{\infty} B(c_n, \tilde{x}_n) \right) \cup \bigcup_{n=1}^{\infty} \partial B(c_n, \tilde{x}_n).$$

Since $\tilde{x} \in l^\infty_+$, it follows that $A(x) \in \mathcal{H}(X; C)$. It is straightforward to check that $D(A(x), A(y)) = \|\tilde{x} - \tilde{y}\| = \|x - y\|$ for all $x, y \in l^\infty_+$.

Every separable metric space embeds isometrically into l^∞ by Frechét's theorem. The map $(x_1, x_2, \dots) \mapsto \sqrt{2}(x_1^+, x_1^-, x_2^+, x_2^-, \dots)$ is easily seen to be a $\sqrt{2}$-bi-Lipschitz embedding of l^∞ into l^∞_+. Also, a bounded subset of l^∞ can be isometrically mapped into l^∞_+ by translation $x \mapsto x + (M, M, \dots)$, where M is sufficiently large.

It remains to show the sharpness of the constant $\sqrt{2}$. Suppose that $F : \mathbb{Z} \to \mathcal{H}(\mathbb{R}; \mathbb{R})$ is an L-bi-Lipschitz embedding. Observe that for any sets $A, B \in \mathcal{H}(\mathbb{R}; \mathbb{R})$ we have $D(A, B) \leq \max\{D(A, \mathbb{R}), D(B, \mathbb{R})\}$. Therefore, for every $n \in \mathbb{Z}$

$$\begin{aligned} 2n/L &\leq D(F(n), F(-n)) \\ &\leq \max\{D(F(n), \mathbb{R}), D(F(-n), \mathbb{R})\} \\ &\leq \max\{D(F(n), F(0)), D(F(-n), F(0))\} + D(F(0), \mathbb{R}) \\ &\leq Ln + D(F(0), \mathbb{R}). \end{aligned}$$

Letting $n \to \infty$, we obtain $L \geq \sqrt{2}$. $\square$

Despite the last statement of Theorem 5.1, many spaces of the form $\mathcal{H}(X; C)$ do contain an isometric copy of l^∞. The following is a partial result in this direction.

PROPOSITION 5.2. *Let X be a geodesic metric space. Suppose that $C \in \mathrm{CL}(X)$ contains a sequence $\{c_n : n \geq 1\}$ such that $\operatorname{dist}(c_n, C \setminus \{c_n\}) \to \infty$ as $n \to \infty$. Then l^∞ isometrically embeds into $\mathcal{H}(X; C)$.*

Proof. For each $n \geq 1$ let $d_n = \operatorname{dist}(c_n, C \setminus \{c_n\})$ and choose a geodesic arc $\gamma_n : [0,1] \to X$ of length $d_n/2$ so that $\gamma(0) = c_n$. For $x = (x_1, x_2, \dots) \in l^\infty$ set $x_{ij} = \min\{d_j/4, \max\{-d_j/4, x_i\}\}$, $i, j \geq 1$. The sequence

$$\tilde{x} := (x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}, \dots)$$

satisfies $|\tilde{x}_n| \leq d_n/4$, and the map $x \mapsto \tilde{x}$ is an isometry of l^∞ into itself.

Let $A(x) = (C \setminus \{c_n : n \geq 1\}) \cup \{\gamma_n(2\tilde{x}_n/d_n + 1/2) : n \geq 1\}$. It is clear that $A(x) \in \mathcal{H}(\mathbb{R}; C)$ and $D(A(x), A(y)) \leq \|\tilde{x} - \tilde{y}\|_{l^\infty}$ for all $x, y \in l^\infty$. To prove the reverse inequality, choose k so that $|\tilde{x}_k - \tilde{y}_k| = \|\tilde{x} - \tilde{y}\|_{l^\infty}$ and observe that

$$\operatorname{dist}(\gamma_n(2\tilde{x}_n/d_n + 1/2), A(y)) = |\tilde{x}_n - \tilde{y}_n|.$$

This completes the proof. □

For example, Proposition 5.2 implies that every separable metric space admits an isometric embedding into $\mathcal{H}(\mathbb{R}, \{n^2 : n \in \mathbb{Z}\})$.

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Department of Mathematics, Mailstop 3368, Texas A&M University, College Station, TX 77843-3368
E-mail address: lkovalev@math.tamu.edu

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61822
E-mail address: tyson@math.uiuc.edu