Quasiregular Gradient Mappings and Strong Solutions of Elliptic Equations

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Abstract. We prove that quasiregular gradient mappings exhibit higher degree of Hölder continuity than the one that is optimal for general quasiregular mappings. This improves a classical result of Morrey on the regularity of strong solutions of uniformly elliptic PDEs with measurable coefficients. Our Hölder estimate for homogeneous solutions of such equations is close to the best possible.

1. Introduction

Let Ω be a domain in $\mathbb{R}^2$ and let $f : \Omega \to \mathbb{R}^2$ be a Sobolev mapping of the class $W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$. For almost every point $z \in \Omega$ the mapping f is differentiable at z, which allows one to define the differential matrix $Df(z)$ and Jacobian determinant $J_f(z) = \det Df(z)$. Let $|A|$ denote the operator norm of a matrix A. The mapping f is called K-quasiregular if $|Df(z)|^2 \le K J_f(z)$ for a.e. $z \in \Omega$, where $K \ge 1$.

Every K-quasiregular mapping f is locally Hölder continuous with exponent $1/K$ (i.e. $f \in C^{0,1/K}_{\mathrm{loc}}(\Omega)$), as was proved by Morrey [**23**] for one-to-one mappings and by Nirenberg [**24**] in the general case. Actually, Morrey and Nirenberg established the Hölder norm estimates assuming $f \in C^1$, but the general case is not much different, see [**14**]. Different proofs were given by Ahlfors [**1**] and Mori [**22**].

It is often convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$ and introduce the complex differential operators

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

A mapping $f \in W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{C})$ is K-quasiregular in Ω if and only if

$$\left|\frac{\partial f}{\partial \bar z}\right| \le \frac{K-1}{K+1}\left|\frac{\partial f}{\partial z}\right| \qquad \text{a.e. in } \Omega.$$

Given a real-valued function $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$, we can consider its complex gradient $f = \partial u/\partial z$ as a mapping from Ω into $\mathbb{C}$. It is easy to see that f is holomorphic if and only if u is harmonic. More generally, f is quasiregular if and only if u is

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a strong solution of a second order uniformly elliptic PDE in non-divergence form with no terms of lower order.

In early 1980s Bojarski and Iwaniec [**7**] observed that the complex gradient of a p-harmonic function (for $p \geq 2$) is a K-quasiregular mapping with $K = p - 1$. This led, among others, to the conclusion that p-harmonic functions are of class $C^{1,\alpha}_{\mathrm{loc}}$ with $\alpha = 1/(p-1)$. This result was later extended to $p > 1$ [**3, 21**], sharpened [**2, 15**] and generalized [**20**].

The above demonstrates that quasiregular gradient mappings (i.e. those of the form $f = \partial u/\partial z$ with $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$) form an important subclass of quasiregular mappings. It is therefore of interest to establish the best possible Hölder estimates for this class. First it should be observed that the $C^{0,1/K}_{\mathrm{loc}}$-continuity of general K-quasiregular mappings cannot be improved, as is demonstrated by the example $\varphi(z) = z|z|^{1/K-1}$. Note also that $\varphi(z) = \partial(c|z|^{1+1/K})/\partial\bar{z}$, where $c = 2K/(K+1)$.

This stands in contrast to the main result of the present paper (Theorem 1.1). Before stating it, let us recall definitions of relevant Hölder spaces. Given a compact set $E \subset \mathbb{C}$ and a continuous function $h : E \to \mathbb{C}$, define the modulus of continuity of h on E by

$$\omega_h(E,\delta) = \sup\left\{|h(z_1) - h(z_2)| : z_1, z_2 \in E, |z_1 - z_2| \leq \delta\right\}.$$

The Hölder space of order α, $0 < \alpha \leq 1$, is

$$C^{0,\alpha}(E) = \left\{h : E \to \mathbb{C} : \sup_{\delta>0} \delta^{-\alpha}\omega_h(E,\delta) < \infty\right\}.$$

We will also use the little Hölder space $c^{0,\alpha}(E)$, $0 < \alpha < 1$:

$$c^{0,\alpha}(E) = \left\{h \in C^{0,\alpha}(E) : \lim_{\delta\to 0} \delta^{-\alpha}\omega_h(E,\delta) = 0\right\}.$$

For $k \geq 1$ the spaces $C^{k,\alpha}(E)$ and $c^{k,\alpha}(E)$ consist of functions whose k-th order partial derivatives are in $C^{0,\alpha}(E)$ or $c^{0,\alpha}(E)$, respectively. Finally, $C^{k,\alpha}_{\mathrm{loc}}(\Omega)$ consists of the functions that belong to $C^{k,\alpha}_{\mathrm{loc}}(E)$ for every compact set $E \subset \Omega$. The spaces $c^{k,\alpha}_{\mathrm{loc}}(\Omega)$ are defined similarly.

THEOREM 1.1. *Let $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$. Suppose that $f = \partial u/\partial z$ is K-quasiregular in Ω, $K > 1$. Then $f \in c^{0,1/K}_{\mathrm{loc}}(\Omega)$ and, consequently, $u \in c^{1,1/K}_{\mathrm{loc}}(\Omega)$.*

In view of this result it seems probable that K-quasiregular gradient mappings are $C^{0,\alpha}_{\mathrm{loc}}$-continuous with $\alpha > 1/K$. This was conjectured by D'Onofrio [**9**]; in fact, his conjecture was the starting point of our investigation. We believe that the best possible value of α is

$$\frac{1}{2}\left(\sqrt{1 + 14K^{-1} + K^{-2}} - 1 - K^{-1}\right),$$

see §§4,5.

The paper is organized as follows. In §2 we discuss a certain rigidity property of quasiregular mappings [**16, 17**] which is crucial for the proof of Theorem 1.1. The proof itself is carried out in §3. An upper bound for the Hölder exponent is given in §4. This bound is realized by a homogeneous mapping, which leads us to investigate such mappings in §5. In the final section our results are restated in terms of uniformly elliptic partial differential equations in non-divergence form.

2. Preliminaries

First we examine the K-quasiregular mappings $\Omega \to \mathbb{C}$ that do not belong to $c^{0,1/K}_{\mathrm{loc}}(\Omega)$. According to the following theorem, they have at least one power-like singularity.

THEOREM 2.1. *Let $f : \Omega \to \mathbb{C}$ be a K-quasiregular mapping, and let E be a compact subset of Ω. If $f \notin c^{0,1/K}(E)$, then there exist $z_0 \in E$, $A > 0$, and a continuous function $\theta : (0, 1) \to \mathbb{R}$ such that:*

(i) *f is one-to-one in a neighborhood of z_0;*
(ii) *as $r \downarrow 0$,*

$$f(z_0 + re^{i\varphi}) = f(z_0) + Ar^{1/K}e^{i(\varphi+\theta(r))} + o(r^{1/K}), \tag{2.1}$$

where $o(r^{1/K})$ is uniform in $\varphi \in [0, 2\pi]$.

A prototype of this result appeared in [**16**], where the asymptotic expansion (2.1) was derived under the assumption that

$$\limsup_{z\to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|^{1/K}} > 0. \tag{2.2}$$

This result is weaker than Theorem 2.1, because it is not obvious that the condition $f \notin c^{0,1/K}(E)$ implies (2.2) for some $z_0 \in E$. However, in a later paper [**17**] it is proved that if a K-quasiconformal mapping f fails to be in $c^{0,1/K}(E)$, then there exists $z_0 \in E$ for which (2.2) holds. (A mapping is K-quasiconformal if it is both K-quasiregular and one-to-one). This reduces the proof of Theorem 2.1 to the following simple argument.

PROOF. By the above-mentioned results Theorem 2.1 is true for one-to-one mappings. Therefore, we only need to prove statement (i).

Suppose that f is K-quasiregular in Ω, and $f \notin c^{0,1/K}(E)$. Then there exist sequences $\{a_j\}$ and $\{b_j\}$ with a common limit $z_0 \in E$ such that $a_j, b_j \in E$ and

$$\lim_{j\to\infty} \frac{|f(a_j) - f(b_j)|}{|a_j - b_j|^{1/K}} > 0. \tag{2.3}$$

The mapping f admits the Stoilow factorization [**19**, p.247] $f = \psi \circ h$, where h is K-quasiconformal and ψ is a holomorphic function.

Suppose that ψ' vanishes at $h(z_0)$. Then for any $\varepsilon > 0$ there is $r > 0$ such that $|\psi'| \le \varepsilon$ in $D_r = \{\zeta \in \mathbb{C} : |\zeta - h(z_0)| < r\}$ and $D_r \subset h(\Omega)$. Choose a compact set $E \subset \Omega$ that contains a neighborhood of z_0, and let

$$B = \sup_{z_1,z_2\in E} \frac{|h(z_1) - h(z_2)|}{|z_1 - z_2|^{1/K}}.$$

For large j we have $a_j, b_j \in E \cap h^{-1}(D_r)$, hence

$$|f(a_j) - f(b_j)| \le \varepsilon|h(a_j) - h(b_j)| \le B\varepsilon|a_j - b_j|^{1/K}.$$

Since $\varepsilon > 0$ was arbitrary, this contradicts (2.3). Therefore, $\psi'(h(z_0)) \ne 0$, which proves statement (i) of the theorem. □

For future references we record a special case of the Poincaré lemma.

LEMMA 2.2. *Suppose that $f \in W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{C})$, where Ω is a domain in $\mathbb{C}$. If Ω is simply connected, then the following statements are equivalent.*

(i) *there exists* $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ *such that* $f = \partial u/\partial z$ *a.e. in* Ω*;*
(ii) $\operatorname{Im} \partial f/\partial \bar{z} = 0$ *a.e. in* Ω.

For general Ω (i) *implies* (ii).

Proof. Let us write $f(x+iy) = \xi(x,y) + i\eta(x,y)$ with x, y, ξ, η real. One immediately sees that (i) is equivalent to the exactness of differential form $\xi dx - \eta dy$. Since

$$\operatorname{Im} \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \operatorname{Im} \left\{ \frac{\partial \xi}{\partial x} + i \frac{\partial \eta}{\partial x} + i \frac{\partial \xi}{\partial y} - \frac{\partial \eta}{\partial y} \right\} = \frac{1}{2} \left\{ \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y} \right\},$$

condition (ii) holds if and only if the form $\xi dx - \eta dy$ is closed. Therefore, (i) implies (ii). When Ω is simply-connected, (i) is equivalent to (ii) by the Poincaré lemma. □

3. Proof of Theorem 1.1

Proof. Let $f : \Omega \to \mathbb{C}$ is a K-quasiregular gradient mapping, i.e. $f = \partial u/\partial z$. By Lemma 2.2

$$\operatorname{Im} \frac{\partial f}{\partial \bar{z}} = 0, \quad \text{a.e. } z \in \Omega. \tag{3.1}$$

Suppose that $f \notin c^{0,1/K}_{\mathrm{loc}}(\Omega)$; this will eventually lead to a contradiction. Let z_0, A and θ be as in Theorem 2.1. Composing f with appropriate linear transformations, we can make sure that $z_0 = f(z_0) = 0$, $A = 1$, and

$$\alpha := \limsup_{r \to 0} \cos \theta(r) > 0. \tag{3.2}$$

Regarding (3.2), note that by Theorem 2.2 in [**16**], $\lim_{r\to 0} \theta(r)$ need not exist. But we can always replace f with $e^{i\psi} f$, where the constant $\psi \in \mathbb{R}$ is chosen so that (3.2) holds.

Let $R > 0$ be sufficiently small so that

$$G := \{re^{i\varphi} : 0 < r < R, 0 < \varphi < \pi/2\} \subset\subset \Omega.$$

Applying Stokes' theorem [**18**, Thm. 1.1.1] and using (3.1), we obtain

$$\operatorname{Re} \int_{\partial G} f(z) dz = \operatorname{Re} \int_G \frac{\partial f}{\partial \bar{z}} \, d\bar{z} \wedge dz = \operatorname{Re} \left\{ (-2i) \int_G \frac{\partial f}{\partial \bar{z}} \, d\mathcal{L}^2 \right\} = 0,$$

where $\mathcal{L}^2$ is the 2-dimensional Lebesgue measure. The above application of Stokes' theorem can be justified as follows. By Theorem 3 in [**10**, 4.2], there exists a sequence $\{f_k\} \subset W^{1,p}(G) \cap C^\infty(\overline{G})$ such that $f_k \to f$ in $W^{1,p}(G)$. In particular,

$$\int_G \frac{\partial f_k}{\partial \bar{z}} \, d\mathcal{L}^2 \to \int_G \frac{\partial f}{\partial \bar{z}} \, d\mathcal{L}^2.$$

Furthermore, $\int_{\partial G} f_k(z) dz \to \int_{\partial G} f(z) dz$ by Theorem 1 in [**10**, 4.3]. Thus we can pass to the limit $k \to \infty$ after applying Stokes' theorem to f_k.

On the other hand, $\int_{\partial G} f(z)dz$ can be computed directly using (2.1).

$$\begin{aligned}\operatorname{Re}\int_{\partial G} f(z)dz &= \operatorname{Re}\int_0^R r^{1/K}e^{i\theta(r)}dr + \operatorname{Re}\int_0^{\pi/2} R^{1/K}e^{i(\varphi+\theta(R))}iRe^{i\varphi}d\varphi \\ &\quad - \operatorname{Re}\int_0^R r^{1/K}e^{i(\pi/2+\theta(r))}idr + o(R^{1+1/K}) \\ &= 2\int_0^R r^{1/K}\cos\theta(r)dr - R^{1+1/K}\int_0^{\pi/2}\sin(2\varphi+\theta(R))d\varphi \\ &\quad + o(R^{1+1/K}), \quad R\to 0.\end{aligned}$$

Since $\int_0^{\pi/2}\sin(2\varphi+\theta(R))d\varphi = \cos\theta(R)$, we conclude that

$$2\int_0^R r^{1/K}\cos\theta(r)dr = R^{1+1/K}\cos\theta(R) + o(R^{1+1/K}), \quad R\to 0. \tag{3.3}$$

Let us introduce new variables $s = r^{1+1/K}$, $S = R^{1+1/K}$, and a function $\tau(s) = \theta(s^{K/(K+1)})$, $s\in(0,1]$. With this notation (3.3) simplifies to

$$\frac{2K}{K+1}\int_0^S \cos\tau(s)ds = S\cos\tau(S) + o(S), \quad S\to 0. \tag{3.4}$$

Suppose that $\cos\tau(s)\to\alpha$ as $s\to 0$. (Recall that $\alpha>0$ is defined by (3.2).) Dividing both sides of (3.4) by S and letting $S\to 0$, we obtain $2K\alpha/(K+1)=\alpha$, a contradiction. Therefore, $\lim_{s\to 0}\cos\tau(s)$ does not exist. This allows us to choose β and γ so that

$$\liminf_{s\to 0}\cos\tau(s) < \gamma < \beta < \alpha \quad\text{and}\quad 0 < \frac{K+1}{2K}\beta < \gamma. \tag{3.5}$$

There is a sequence s_j such that $s_j\downarrow 0$ and $\cos\tau(s_j)=\beta$ for all j. Since τ is continuous, for sufficiently large j we can define $t_j = \inf\{t : t > s_j, \cos\tau(t)=\gamma\}$. Our choice of γ implies $t_j\downarrow 0$ as $j\to\infty$.

Now by (3.4) we have

$$\int_0^{s_j}\cos\tau(s)ds = \frac{K+1}{2K}\beta s_j + o(s_j) \tag{3.6}$$

The definition of t_j and (3.5) yield

$$\int_{s_j}^{t_j}\cos\tau(s)ds \ge \gamma(t_j - s_j) \ge \frac{K+1}{2K}\beta(t_j-s_j). \tag{3.7}$$

Adding (3.6) to (3.7) we obtain

$$\int_0^{t_j}\cos\tau(s)ds \ge \frac{K+1}{2K}\beta t_j + o(s_j) = \frac{K+1}{2K}\beta t_j + o(t_j). \tag{3.8}$$

But (3.4) implies

$$\int_0^{t_j}\cos\tau(s)ds = \frac{K+1}{2K}\gamma t_j + o(t_j) \tag{3.9}$$

Comparing (3.8) with (3.9) we conclude that $\gamma\ge\beta$, a contradiction. This completes the proof of Theorem 1.1. □

4. An example

The following example of a K-quasiregular gradient mapping is essentially the same as in section 7 of [**15**]. The only difference is that parameter α in (4.2) is less than 1, while in [**15**] it is greater than 1.

EXAMPLE 4.1. Given $K > 1$, let $\alpha = \alpha_K$, where

$$\alpha_K = \frac{1}{2}\left(\sqrt{1 + 14K^{-1} + K^{-2}} - 1 - K^{-1}\right), \tag{4.1}$$

and let

$$f(z) = |z|^\alpha \left(\frac{z}{|z|} - \frac{1-\alpha}{3+\alpha}\frac{|z|^3}{z^3}\right), \quad z \in \mathbb{C}. \tag{4.2}$$

Then f is a K-quasiregular gradient mapping in $\mathbb{C}$, and $f \notin c^{0,\alpha}_{\mathrm{loc}}(\mathbb{C})$.

PROOF. Elementary computations show that $0 < \alpha < 1$ (see Remark 4.2 below). It is obvious that $\limsup_{z\to 0}|f(z)|/|z|^\alpha > 0$, hence $f \notin c^{0,\alpha}_{\mathrm{loc}}(\mathbb{C})$. In a neighborhood of any point $z \neq 0$ one can write (4.2) as

$$f(z) = z^{\frac{\alpha+1}{2}}\bar{z}^{\frac{\alpha-1}{2}} - \frac{1-\alpha}{3+\alpha} z^{\frac{\alpha-3}{2}}\bar{z}^{\frac{\alpha+3}{2}}.$$

Therefore,

$$\begin{aligned}
\frac{\partial f}{\partial z} &= \frac{\alpha+1}{2}|z|^{\alpha-1} + \frac{(1-\alpha)(3-\alpha)}{2(3+\alpha)} z^{\frac{\alpha-5}{2}}\bar{z}^{\frac{\alpha+3}{2}};\\
\frac{\partial f}{\partial \bar{z}} &= \frac{\alpha-1}{2} z^{\frac{\alpha+1}{2}}\bar{z}^{\frac{\alpha-3}{2}} - \frac{1-\alpha}{2} z^{\frac{\alpha-3}{2}}\bar{z}^{\frac{\alpha+1}{2}}\\
&= (\alpha-1)|z|^{\alpha-3}\,\mathrm{Re}(z^2).
\end{aligned}$$

Since the first derivatives of f are homogeneous of degree $\alpha - 1 > -1$, it follows that $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C};\mathbb{C})$. By Lemma 2.2 f is the complex gradient of some function $u \in W^{2,2}_{\mathrm{loc}}(\mathbb{C})$; in fact, one can take

$$u(z) = \frac{4}{\alpha+3}|z|^{\alpha-1}\,\mathrm{Re}(z^2). \tag{4.3}$$

It remains to show that f is K-quasiregular. In view of homogeneity it suffices to prove that

$$\left|\frac{\partial f}{\partial \bar{z}}\right| \le \frac{K-1}{K+1}\left|\frac{\partial f}{\partial z}\right|, \quad z = e^{i\varphi},\ 0 \le \varphi \le 2\pi. \tag{4.4}$$

When $z = e^{i\varphi}$, we have

$$\begin{aligned}
\left|\frac{\partial f}{\partial z}\right|^2 &= \frac{1}{4}\left|\alpha + 1 + \frac{(1-\alpha)(3-\alpha)}{3+\alpha}e^{-4i\varphi}\right|^2\\
&= \frac{1}{4}\left\{(\alpha+1)^2 + \frac{(1-\alpha)^2(3-\alpha)^2}{(3+\alpha)^2} + 2\frac{(1-\alpha^2)(3-\alpha)}{3+\alpha}\cos 4\varphi\right\}\\
&= \frac{1}{4}\left\{\frac{64\alpha^2}{(3+\alpha)^2} + 4\frac{(1-\alpha^2)(3-\alpha)}{3+\alpha}\cos^2 2\varphi\right\}\\
&= \frac{16\alpha^2}{(3+\alpha)^2} + \frac{(1-\alpha^2)(3-\alpha)}{3+\alpha}\cos^2 2\varphi.
\end{aligned}$$

Since

$$\left|\frac{\partial f}{\partial \bar z}\right|^2 = (1-\alpha)^2\cos^2 2\varphi,$$

it follows that

$$\begin{aligned}\frac{|\partial f/\partial z|^2}{|\partial f/\partial \bar z|^2} &= \frac{16\alpha^2}{(3+\alpha)^2(1-\alpha)^2\cos^2 2\varphi} + \frac{(1-\alpha^2)(3-\alpha)}{(1-\alpha)^2(3+\alpha)} \\ &\ge \frac{16\alpha^2}{(3+\alpha)^2(1-\alpha)^2} + \frac{(1-\alpha^2)(9-\alpha^2)}{(1-\alpha)^2(3+\alpha)^2} \\ &= \frac{(3+\alpha^2)^2}{(1-\alpha)^2(3+\alpha)^2}.\end{aligned} \tag{4.5}$$

In order to prove (4.4), it remains to verify the following identity.

$$\frac{(1-\alpha)(3+\alpha)}{3+\alpha^2} = \frac{K-1}{K+1}. \tag{4.6}$$

But (4.6) is equivalent to quadratic equation $\alpha^2 + (1+K^{-1})\alpha - 3K^{-1} = 0$, whose positive root is $\alpha = \alpha_K$. □

Remark 4.2. For every $K > 1$ we have

$$0 < \alpha_K < \min\{1/\sqrt{K}, 3/K\}.$$

Indeed, $\alpha_K > 0$ because

$$\sqrt{1+14K^{-1}+K^{-2}} > \sqrt{1+2K^{-1}+K^{-2}} = 1+K^{-1}.$$

With notation $t = 1/\sqrt{K}$ the inequality $\alpha_K < 1/\sqrt{K}$ takes form

$$\frac{1}{2}\left(\sqrt{1+14t^2+t^4} - 1 - t^2\right) < t, \quad 0 < t < 1,$$

which is equivalent to $1+14t^2+t^4 < (1+t)^4$. After further simplifications it reduces to the trivial inequality $4t(t-1)^2 > 0$.

It remains to prove that $\alpha_K < 3/K$, or, equivalently,

$$\sqrt{1+14K^{-1}+K^{-2}} < 1+7K^{-1}.$$

The latter inequality is verified by squaring both sides.

Conjecture 4.3. Every K-quasiregular gradient mapping in dimension 2 is locally Hölder continuous with exponent α_K.

We will return to Example 4.1 in §6.

5. Homogeneous mappings

It is natural to attempt to prove Conjecture 4.3 for homogeneous mappings first, since the conjectural extremal (Example 4.1) is homogeneous. The following theorem applies to more general mappings than homogeneous ones, and provides a nearly optimal Hölder exponent.

Theorem 5.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a K-quasiregular gradient mapping, $K > 1$, and suppose that*

$$f(re^{i\varphi}) = g(r)h(\varphi), \quad r \ge 0,\ \varphi \in \mathbb{R}, \tag{5.1}$$

where $g : [0,\infty) \to [0,\infty)$ *is increasing and* $h : \mathbb{R} \to \mathbb{C}$ *has period* 2π. *Then* $f \in C^{0,\beta}_{\mathrm{loc}}(\mathbb{C})$, *where* $\beta = \beta_K \in (0,1)$ *is determined from the equation*

$$\frac{(1-\beta)(3+\beta)}{\sqrt{9+22\beta^2+\beta^4}} = \frac{K-1}{K+1}. \tag{5.2}$$

Proof. To avoid trivialities, assume f nonconstant. This implies $|h(\varphi)| > 0$ for all φ; otherwise the set $f^{-1}(0)$ would not be discrete. Choose $r > 0$ so that $g'(r)$ exists and the circle $\{z : |z| = r\}$ contains no branch points of f (which are isolated by the Stoilow factorization theorem). Next we use the fact that all directional derivatives of a quasiconformal mapping are of comparable size, see [**19**, II.9]. More precisely,

$$\begin{aligned} g(r)\limsup_{t\to 0}\frac{|h(\varphi+t)-h(\varphi)|}{|t|} &= \limsup_{t\to 0}\frac{|f(re^{i(\varphi+t)})-f(re^{i\varphi})|}{|t|} \\ &\le Cr\limsup_{t\to 0}\frac{|f((r+t)e^{i\varphi})-f(re^{i\varphi})|}{|t|} \\ &= Crg'(r)|h(\varphi)|, \quad \varphi\in\mathbb{R}, \end{aligned}$$

where C is an absolute constant. Thus $h \in C^{0,1}(\mathbb{R})$. Let L be the Lipschitz constant of h. Fix arbitrary $\varphi \in \mathbb{R}$ and $r \in (0,\infty)$, and apply [**19**, II.9] to majorize the radial derivative of f by the tangential one.

$$\begin{aligned} \limsup_{t\to 0}\frac{|g(r+t)-g(r)|}{|t|} &= \lim_{t\to 0}\frac{|f((r+t)e^{i\varphi})-f(re^{i\varphi})|}{|th(\varphi)|} \\ &\le \frac{C}{|h(\varphi)|}\limsup_{t\to 0}\frac{|f(re^{i(\varphi+t)})-f(re^{i\varphi})|}{r|t|} \le \frac{CLg(r)}{r|h(\varphi)|}. \end{aligned} \tag{5.3}$$

Therefore, $g \in C^{0,1}(a,b)$ whenever $0 < a < b < \infty$. So far we have demonstrated that $f \in C^{0,1}_{\mathrm{loc}}(\mathbb{C}\setminus\{0\})$.

We are going to prove that

$$\limsup_{t\to 0} r^{-\beta} g(r) < \infty, \tag{5.4}$$

but first let us show that (5.4) implies the conclusion of the theorem. Indeed, from (5.4) and (5.3) follows the existence of a constant C_1 such that $g'(r) \le C_1 r^{\beta-1}$ for a.e. $r \in (0,1)$. Now for every $0 \le r_1 < r_2 \le 1$ we have

$$g(r_2)-g(r_1) = \int_{r_1}^{r_2} g'(r)dr \le C_1\beta^{-1}(r_2^\beta - r_1^\beta) \le C_1\beta^{-1}(r_2-r_1)^\beta.$$

Consequently, for every $\varphi_1, \varphi_2 \in \mathbb{R}$ with $|\varphi_1-\varphi_2| \le \pi$

$$\begin{aligned} |f(r_1e^{i\varphi_1})-f(r_2e^{i\varphi_2})| &\le |f(r_1e^{i\varphi_1})-f(r_2e^{i\varphi_1})| + |f(r_2e^{i\varphi_1})-f(r_2e^{i\varphi_2})| \\ &\le C_1\beta^{-1}|r_1-r_2|^\beta \max_\varphi |h(\varphi)| + g(r_2)L|\varphi_1-\varphi_2|. \end{aligned}$$

It is easy to see that both terms are dominated by $|r_1e^{i\varphi_1} - r_2e^{i\varphi_2}|^\beta$, which implies $f \in C^{0,\beta}_{\mathrm{loc}}(\mathbb{C})$.

It remains to prove (5.4). Since h is absolutely continuous, its Fourier series $\sum_{n\in\mathbb{Z}} c_n e^{in\varphi}$ converges to h uniformly [**28**, II.8.6]. Thus

$$f(re^{i\varphi}) = g(r)\sum_{n\in\mathbb{Z}} c_n e^{in\varphi}, \quad r \ge 0,\ \varphi\in\mathbb{R}. \tag{5.5}$$

Furthermore, $h \in W^{1,2}(\mathbb{R};\mathbb{C})$ because $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C};\mathbb{C})$. We can apply the differential operators

$$\frac{\partial}{\partial z} = \frac{e^{-i\varphi}}{2}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \varphi}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{e^{i\varphi}}{2}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \varphi}\right).$$

to (5.5) provided that $g'(r)$ exists. Doing so we obtain

$$\frac{\partial f}{\partial z} = \frac{g(r)}{2r}\sum_{n\in\mathbb{Z}}(rg'(r)/g(r) + n)c_n e^{i(n-1)\varphi} \tag{5.6}$$

and

$$\frac{\partial f}{\partial \bar{z}} = \frac{g(r)}{2r}\sum_{n\in\mathbb{Z}}(rg'(r)/g(r) - n)c_n e^{i(n+1)\varphi}, \tag{5.7}$$

where both series converge in L^2. Since f is K-quasiregular,

$$\left|\frac{\partial f}{\partial \bar{z}}\right| \le k\left|\frac{\partial f}{\partial z}\right| \qquad \text{a.e., where } k = \frac{K-1}{K+1}. \tag{5.8}$$

In view of (5.6)-(5.8) and Parseval's formula we have for a.e. r

$$\sum_{n\in\mathbb{Z}}(\gamma - n)^2|c_n|^2 \le k^2\sum_{n\in\mathbb{Z}}(\gamma+n)^2|c_n|^2, \quad \gamma = rg'(r)/g(r). \tag{5.9}$$

Our goal is to prove that $\gamma \ge \beta$; in doing so we may assume that $\gamma < 1$. Recall that $\partial f/\partial\bar{z}$ is real-valued by Lemma 2.2. By virtue of (5.7) this is equivalent to

$$(\gamma - n)c_n = (\gamma + n + 2)\bar{c}_{-n-2}, \quad n \in \mathbb{Z}. \tag{5.10}$$

Combining (5.9) and (5.10), we obtain

$$(\gamma+1)^2|c_{-1}|^2 + 2\sum_{n=0}^{\infty}(\gamma-n)^2|c_n|^2 \le k^2(\gamma-1)^2|c_{-1}|^2 + k^2\sum_{n=0}^{\infty}\left\{(\gamma+n)^2 + \left(\frac{n+2-\gamma}{n+2+\gamma}\right)^2(\gamma-n)^2\right\}|c_n|^2.$$

Since $(\gamma+1)^2 > k^2(\gamma-1)^2$, it follows that

$$\sum_{n=0}^{\infty}\left\{\left(2 - k^2\left(\frac{n+2-\gamma}{n+2+\gamma}\right)^2\right)(\gamma-n)^2 - k^2(\gamma+n)^2\right\}|c_n|^2 \le 0,$$

where the equality can hold only if $c_{-1} = 0$. Since f is nonconstant, at least one of the coefficients c_n is nonzero, hence

$$\left(2 - k^2\left(\frac{n+2-\gamma}{n+2+\gamma}\right)^2\right)(\gamma-n)^2 - k^2(\gamma+n)^2 \le 0 \tag{5.11}$$

for some $n \ge 0$. When $n = 0$, inequality (5.11) does not hold since

$$\left(2 - k^2\left(\frac{2-\gamma}{2+\gamma}\right)^2\right)\gamma^2 - k^2\gamma^2 > (2-k^2)\gamma^2 - k^2\gamma^2 = 2(1-k^2)\gamma^2 > 0.$$

Therefore, (5.11) must hold for some $n \ge 1$. Dividing (5.11) by $(\gamma - n)^2$ and rearranging terms, we obtain

$$\frac{2}{k^2} \le \left(\frac{n+\gamma}{n-\gamma}\right)^2 + \left(\frac{n+2-\gamma}{n+2+\gamma}\right)^2. \tag{5.12}$$

As a function of n, the right-hand side of (5.12) decreases for $n \in [1, \infty)$, because its derivative is equal to

$$-\frac{16\gamma(n+1)(n(n+2)+2\gamma(n+1)^2+3\gamma^2+2\gamma^3)}{(n-\gamma)^3(n+2+\gamma)^3} < 0.$$

Thus (5.12) holds with $n = 1$.

$$\frac{1}{k^2} \le \frac{1}{2}\left\{\left(\frac{1+\gamma}{1-\gamma}\right)^2 + \left(\frac{3-\gamma}{3+\gamma}\right)^2\right\} = \frac{9+22\gamma^2+\gamma^4}{(1-\gamma)^2(3+\gamma)^2}. \tag{5.13}$$

Let $\tau(\gamma)$ denote the right-hand side of (5.13). Differentiation yields

$$\tau'(\gamma) = \frac{4(3+\gamma^2)(3+\gamma(14-\gamma))}{(1-\gamma)^3(3+\gamma)^3} > 0, \quad 0 < \gamma < 1.$$

Therefore, $1/\sqrt{\tau(\gamma)}$ strictly decreases from 1 to 0 on the interval $(0, 1)$. It follows that β is well-defined by (5.2) and $\gamma \ge \beta$. Integrating the differential inequality $g'(r)/g(r) \ge \beta/r$, we obtain (5.4). □

Two remarks are in order.

Remark 5.2. Comparing (5.2) and (4.6), we see that $\beta_K < \alpha_K$ for all $K > 1$. This was to be expected, because we passed from pointwise inequality (5.8) to the L^2 estimate (5.9). On the other hand, it is not hard to prove that $\beta_K > 1/K$ for all $K > 1$. Moreover, numerical computations show that the difference $\alpha_K - \beta_K$ does not exceed 0.074, which is a relatively small error. This can be viewed as an evidence in favor of Conjecture 4.3.

Remark 5.3. The proof of Theorem 5.1 is local in nature. If f admits decomposition (5.1) only in a disk centered at 0, then f is locally C^{0,β_K}-continuous in that disk.

In conclusion of this section we briefly indicate an application of Theorem 5.1 to more general mappings. Suppose that $f : \Omega \to \mathbb{C}$ is a K-quasiregular gradient mapping with simple infinitesimal space at some point $z_0 \in \Omega$ (see [**13**] for definitions). Let f_0 be the only element of the infinitesimal space of f at z_0. Then f_0 is a K-quasiregular gradient mapping as well, and it is homogeneous by [**13**, 4.1]. Applying Theorem 5.1 to f_0 and using [**13**, 4.7], we obtain

$$\limsup_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|^\alpha} = 0, \quad 0 < \alpha < \beta_K.$$

6. Elliptic equations

Let Ω be a domain in $\mathbb{R}^2$. Consider the following second-order equation in non-divergence form:

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} = 0, \tag{6.1}$$

where a, b and c are measurable real-valued functions in Ω. Let $\lambda_{1,2}(x,y)$ be the eigenvalues of the coefficient matrix

$$\begin{pmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{pmatrix}.$$

In what follows we assume that the operator appearing in (6.1) is uniformly elliptic, that is, there exist $K > 1$ and $\lambda_0 > 0$ such that

$$\lambda_0 \le \lambda_{1,2}(x, y) \le K\lambda_0 \tag{6.2}$$

for a.e. $(x, y) \in \Omega$.

A function $u \in W^{2,2}_{\text{loc}}(\Omega)$ is called a strong solution of equation (6.1) if it satisfies (6.1) a.e. in Ω. Although a strong solution might not be twice continuously differentiable, it has a continuous representative which belongs to $C^{1,1/K}_{\text{loc}}(\Omega)$. This result, which has no analogue in higher dimensions [**27**], goes back to the classical paper of Morrey [**23**] (see also [**14, 24**] and Chapter 12 of [**11**]). Its proof involves the mapping $(x, y) \mapsto (u_x, -u_y)$, which turns out to be K-quasiregular. Conversely, if a function $u \in W^{2,2}_{\text{loc}}(\Omega)$ is such that $(x, y) \mapsto (u_x, -u_y)$ is K-quasiregular, then one can find $a, b, c \in L^\infty(\Omega)$ such that (6.1) and (6.2) hold [**21**]. (Note that a K-quasiregular mapping in the sense of [**11, 21**] is $(K + \sqrt{K^2 - 1})$-quasiregular according to the definition used in the present paper).

Therefore, Theorem 1.1 can be interpreted in terms of uniformly elliptic equations as follows.

Corollary 6.1. *Suppose that a function* $u \in W^{2,2}_{\text{loc}}(\Omega)$ *solves equation* (6.1) *whose coefficients satisfy* (6.2). *Then* $u \in c^{1,1/K}_{\text{loc}}(\Omega)$.

By virtue of Astala's theorem [**4**, 1.6] the assumption $u \in W^{2,2}_{\text{loc}}(\Omega)$ in Corollary 6.1 can be replaced with $u \in W^{2,q}_{\text{loc}}(\Omega)$, $q > 2K/(K + 1)$. (Actually, q can be taken equal to $2K/(K + 1)$, since Petermichl and Volberg [**25**] have proved the borderline regularity result for Beltrami operators that was conjectured in [**5**].) By [**4**, 1.2] we then have $u \in W^{2,p}_{\text{loc}}(\Omega)$ for any $p < 2K/(K - 1)$. It it plausible that the second derivatives of functions considered in Corollary 6.1 possess even higher degree of integrability. Indeed, under the additional assumption of homogeneity we have $u \in W^{2,p}_{\text{loc}}(\mathbb{C})$ for all $p < 2/(1 - \beta_K)$ by Theorem 5.1.

Example 4.1 shows that the exponent $1/K$ in Corollary 6.1 cannot be replaced with α_K. This negative result also follows from Theorem 3.1 in [**8**] which implies that the function u in (4.3) is a strong solution of equation (6.1) with the coefficient matrix

$$A_K = \begin{pmatrix} 1 + (K-1)\dfrac{x^2}{x^2 + y^2} & (K-1)\dfrac{xy}{x^2 + y^2} \\ (K-1)\dfrac{xy}{x^2 + y^2} & 1 + (K-1)\dfrac{y^2}{x^2 + y^2} \end{pmatrix}.$$

Note that the eigenvalues of A_K are K and 1, hence their ratio is equal to K for all $(x, y) \ne (0, 0)$. At the same time, Example 4.1 and Theorem 3 in [**21**] show that u also satisfies (6.1) with a different coefficients matrix, namely

$$B_K = \begin{pmatrix} 1 - \operatorname{Re}\mu & \operatorname{Im}\mu \\ \operatorname{Im}\mu & 1 + \operatorname{Re}\mu \end{pmatrix}, \quad \mu = \frac{\partial f/\partial \bar{z}}{\partial f/\partial z},$$

where f is defined by (4.2). It is easy to see that the ratio of the eigenvalues of B_K is equal to $(1 + |\mu|)/(1 - |\mu|)$. By (4.5)–(4.6) this ratio is bounded by K and is strictly less than K unless $xy = 0$.

D'Onofrio and Greco [**8**] proved that every strong solution of (6.1) with the coefficient matrix A_K is locally in C^{1,α_K}, where α_K is defined by (4.1). This result supports Conjecture 4.3 which says that the same conclusion should hold whenever the coefficient matrix of (6.1) satisfies (6.2). The proof in [**8**] is based on the analysis of homogeneous solutions of (6.1). In this regard we observe that by Theorem 5.1 every homogeneous solution of (6.1) with an arbitrary coefficient matrix satisfying (6.2) belongs to $C^{1,\beta_K}_{\mathrm{loc}}(\mathbb{R}^2)$, the result being close to the best possible.

Finally, a remark on elliptic equations in divergence form

$$(a(x,y)u_x)_x + (b(x,y)u_y)_x + (b(x,y)u_x)_y + (c(x,y)u_y)_y = 0, \tag{6.3}$$

where a, b and c are as above. Piccinini and Spagnolo [**26**] proved that any weak ($W^{1,2}_{\mathrm{loc}}$) solution of (6.3) belongs to $C^{0,1/\sqrt{K}}_{\mathrm{loc}}(\Omega)$. The Hölder exponent $1/\sqrt{K}$ is sharp (see [**12**] for more on the question of sharpness). Since $\alpha_K < 1/\sqrt{K}$ by Remark 4.2, a strong solution of (6.1) does not necessarily belong to $C^{1,1/\sqrt{K}}_{\mathrm{loc}}(\Omega)$, in contrast with the theorem of Piccinini and Spagnolo.

7. Concluding remarks

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Since this paper was submitted, a counterexample to Conjecture 4.3 has been found [**6**]. On the other hand, it has been confirmed [**6**] that for every $K > 1$ all K-quasiregular gradient mappings are locally Hölder continuous with an exponent strictly greater than $1/K$.

References

[1] L.V. Ahlfors, *On quasiconformal mappings*, J. Analyse Math. **3** (1954), 1–58.
[2] G. Aronsson, *Representation of a p-harmonic function near a critical point in the plane*, Manuscripta Math. **66** (1989), 73–95.
[3] G. Aronsson and P. Lindqvist, *On p-harmonic functions in the plane and their stream functions*, J. Differential Equations **74** (1988), no. 1, 157–178.
[4] K. Astala, *Area distortion of quasiconformal mappings*, Acta Math. **173** (1994), no. 1, 37–60.
[5] K. Astala, T. Iwaniec and E. Saksman, *Beltrami operators in the plane*, Duke Math. J. **107** (2001), no. 1, 27–56.
[6] A. Baernstein II and L.V. Kovalev, *Elliptic equations in two dimensions*, in preparation.
[7] B. Bojarski and T. Iwaniec, *p-harmonic equation and quasiregular mappings*, Partial differential equations (Warsaw, 1984), 25–38, Banach Center Publ., vol. 19, PWN, Warsaw, 1987.
[8] L. D'Onofrio and L. Greco, *On the regularity of solutions to a nonvariational elliptic equation*, Ann. Fac. Sci. Toulouse Math. (6) **11** (2002), no. 1, 47–56.
[9] L. D'Onofrio, *private communication*, October 2003.
[10] L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton, 1992.
[11] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed. Springer-Verlag, Berlin-Heidelberg-New York, 1983.
[12] L. Greco and C. Sbordone, *Sharp upper bounds for the degree of regularity of the solutions to an elliptic equation*, Comm. Partial Differential Equations **27** (2002), no. 5-6, 945–952.
[13] V.Ya. Gutlyanskii, O. Martio, V.I. Ryazanov and M. Vuorinen, *Infinitesimal geometry of quasiregular mappings*, Ann. Acad. Sci. Fenn. Math. **25** (2000), 101–130.
[14] R. Finn and J. Serrin, *On the Hölder continuity of quasiconformal and elliptic mappings*, Trans. Amer. Math. Soc. **89** (1958), 1–15.
[15] T. Iwaniec and J.J. Manfredi, *Regularity of p-harmonic functions on the plane*, Rev. Mat. Iberoamericana **5** (1989), no.1–2, 1–19.
[16] L.V. Kovalev, *Quasiregular mappings of maximal local modulus of continuity*, to appear in Ann. Acad. Sci. Fenn. Math **29** (2004).

[17] ———, *Hölder spaces of quasiconformal mappings*, submitted to Publ. Inst. Math. (Beograd).

[18] S.G. Krantz, *Function theory of several complex variables*, 2nd ed. AMS, Providence, 2001.

[19] O. Lehto, K.I. Virtanen, *Quasiconformal mappings in the plane*, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[20] W.B. Liu and J.W. Barrett, *Higher-order regularity for the solutions of some degenerate quasilinear elliptic equations in the plane*, SIAM J. Math. Anal. **24** (1993), no. 6, 1522–1536.

[21] J.J. Manfredi, *p-harmonic functions in the plane*, Proc. Amer. Math. Soc. **103** (1988), no. 2, 473–479.

[22] A. Mori, *On an absolute constant in the theory of quasi-conformal mappings*, J. Math. Soc. Japan **8** (1956), 156–166.

[23] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc **43** (1938), 126–166.

[24] L. Nirenberg, *On nonlinear elliptic partial differential equations and Hölder continuity*, Comm. Pure Appl. Math. **6** (1953), 103–156.

[25] S. Petermichl and A. Volberg, *Heating of the Ahlfors-Beurling operator: weakly quasiregular maps on the plane are quasiregular*, Duke Math. J. **112** (2002), no. 2, 281–305.

[26] L.C. Piccinini and S. Spagnolo, *On the Hölder continuity of solutions of second order elliptic equations in two variables*, Ann. Scuola Norm. Sup. Pisa (3) **26** (1972), 391–402.

[27] M.V. Safonov, *Unimprovability of estimates of Hölder constants for solutions of linear elliptic equations with measurable coefficients*, Math. USSR-Sb. **60** (1988), no. 1, 269–281.

[28] A. Zigmund, *Trigonometric Series*, 3rd ed., Cambridge Univ. Press, 2003.

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