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Convex functions and quasiconformal mappings

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Abstract. Continuing our investigation of quasiconformal mappings with convex potentials, we obtain a new characterization of quasiuniformly convex functions and improve our earlier results on the existence of quasiconformal mappings with prescribed sets of singularities.

1. Introduction

The purpose of this paper is to improve some of our recent results [**17**] concerning the relation between quasiconformal mappings and convex functions on Euclidean spaces. Given a differentiable convex function $u : \mathbb{R}^n \to \mathbb{R}$, one can consider its gradient ∇u as a mapping from $\mathbb{R}^n$ into itself. The studies of geometric and analytic properties of ∇u are motivated in part by optimal transportation problems with quadratic cost [**27**]. Quasiconformality, being a geometric and analytic property at the same time [**26**], is a natural object for such studies. Recall that a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ ($n \geq 2$) is called *quasiconformal* if f belongs to the Sobolev class $W^{1,n}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ and

$$\|Df(x)\|^n \leq K \det Df(x), \quad \text{a.e. } x \in \mathbb{R}^n, \tag{1.1}$$

for some constant $K \geq 1$. Here and in what follows $\|\cdot\|$ stands for the operator norm of a matrix. We use single bars $|\cdot|$ to denote a norm induced by an inner product.

A direct application of the above definition to ∇u requires one to check that $u \in W^{2,n}_{\text{loc}}(\mathbb{R}^n)$, which can be rather difficult. However, one can prove the quasiconformality of ∇u without ever computing the second derivatives of u [**17**]. The $W^{2,n}_{\text{loc}}$-regularity of u is then obtained as a corollary. In section 2 of this paper we present a new criterion for the quasiconformality of ∇u (Theorem 2.3). Unlike the criteria in [**17**], it does not involve the gradient (or subgradients) of u, but only the values of u itself.

The Jacobian determinant of a quasiconformal mapping ∇u can have a large set of singularities, see [**17**] and section 4 of the present paper. For this reason, such mappings can shed some light on the *quasiconformal Jacobian problem* [**3, 4, 6**].

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In [**17**] the existence of ∇u with prescribed singularities of the Jacobian is proved by a somewhat involved compactness argument. Theorem 3.2 of this paper provides an alternative, more transparent, construction. In section 4 this theorem is applied to the problem of bi-Lipschitz uniformization of quasiconformal mappings in the plane.

2. Quasiuniformly convex functions

Let $\mathcal{H}$ be a Hilbert space over real scalars. We do not assume that $\mathcal{H}$ is infinite-dimensional. Let $u : \mathcal{H} \to \mathbb{R}$ be a continuous convex function. The *subdifferential* of u at a point $z \in \mathcal{H}$ is the set

$$\partial u(z) = \{p \in \mathcal{H} : u(x) \geq u(z) + \langle p, x - z\rangle \ \forall x \in \mathcal{H}\}.$$

The function u is Gâteaux differentiable at $z \in \mathcal{H}$ if and only if $\partial u(z)$ consists of only one vector, denoted $\nabla u(z)$. For $z \in \mathcal{H}$ and $p \in \partial u(z)$ let

$$u_{z,p}(x) = u(x) - u(z) - \langle p, x - z\rangle, \quad x \in \mathcal{H}.$$

The *section* [**5, 12**] of u with the center $z \in \mathcal{H}$, direction $p \in \partial u(z)$, and height $t > 0$ is defined as

$$S_u(z, p, t) = \{x \in \mathcal{H} : u_{z,p}(x) < t\}.$$

If u is Gâteaux differentiable at z, then we write u_z instead of $u_{z,\nabla u(z)}$. We write $B(x, r)$ to denote an open ball with center x and radius r.

Definition 2.1. [**17**] A continuous convex function $u : \mathcal{H} \to \mathbb{R}$ has *round sections* if there exists a constant $\tau \in (0, 1)$ for which the following holds: for every $z \in \mathcal{H}$, $p \in \partial u(z)$ and $t > 0$ there exists $R > 0$ such that

$$B(z, \tau R) \subset S_u(z, p, t) \subset B(z, R). \tag{2.1}$$

The property (2.1) admits several equivalent characterizations, some of which are listed below. We require one more definition [**25**]: a homeomorphism $f : \mathcal{H} \to \mathcal{H}$ is *quasisymmetric*, or η-quasisymmetric, if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$\frac{|f(x) - f(z)|}{|f(y) - f(z)|} \leq \eta\left(\frac{|x - z|}{|y - z|}\right), \quad z \in \mathcal{H},\ x, y \in \mathcal{H} \setminus \{z\}. \tag{2.2}$$

When $1 < \dim \mathcal{H} < \infty$, the classes of quasiconformal and quasisymmetric mappings of $\mathcal{H}$ coincide [**13**].

Theorem 2.2. [**17**], [**15**] *Let $u : \mathcal{H} \to \mathbb{R}$ be a continuous convex function. If $\dim \mathcal{H} > 1$, then all of the following properties are equivalent; if $\dim \mathcal{H} = 1$, then* (i)–(iii) *are equivalent.*

(i) u is Gâteaux differentiable and $\nabla u : \mathcal{H} \to \mathcal{H}$ is η-quasisymmetric for some η;
(ii) u is Gâteaux differentiable but not affine; in addition, there exists $H < \infty$ such that

$$\max_{|x-z|=r} u_z(x) \leq H \min_{|x-z|=r} u_z(x), \quad z \in \mathcal{H},\ r > 0; \tag{2.3}$$

(iii) u has round sections;
(iv) u is Gâteaux differentiable but not affine; in addition, there exists $\delta > 0$ such that

$$\langle \nabla u(x) - \nabla u(y), x - y\rangle \geq \delta |\nabla u(x) - \nabla u(y)||x - y|, \quad x, y \in \mathcal{H}. \tag{2.4}$$

If any of the above holds, then u is Fréchet differentiable. Furthermore, the equivalence is quantitative in the sense that η, H, τ, and δ depend only on one another and not on u or $\mathcal{H}$.

The functions for which the above listed properties hold are called *quasiuniformly (q.u.) convex* in [**17**]. Related classes of convex functions have been studied in [**2, 9, 10**]. All of the previously known criteria for quasiuniform convexity involve (sub)differentials, which sometimes makes them difficult to verify. Theorem 2.3 below gives the first characterization of q.u. convex functions in terms of their values alone. In order to state it, we introduce the second order difference of u at $x \in \mathcal{H}$ with step $h \in \mathcal{H}$:

$$\Delta^2 u(x;h) = u(x+h) - 2u(x) + u(x-h).$$

If u is convex, then $\Delta^2 u(x;h) \geq 0$ for any x and h. Also, $\Delta^2 u_{z,p} = \Delta^2 u$ for any $z \in \mathcal{H}$ and $p \in \partial u(z)$.

THEOREM 2.3. *Let $u : \mathcal{H} \to \mathbb{R}$ be a continuous convex function,* $\dim \mathcal{H} \geq 2$. *Suppose that u is not an affine function. Then u is quasiuniformly convex if and only if there exists $L \geq 1$ such that for any $x \in \mathcal{H}$*

$$\Delta^2 u(x;h_1) \leq L \Delta^2 u(x;h_2) \tag{2.5}$$

whenever $|h_1| = |h_2|$. The equivalence is quantitative in the sense of Theorem 2.2.

PROOF. If u is q.u. convex, then by (2.3) we have

$$\begin{aligned}\Delta^2 u(x;h_1) = \Delta^2 u_x(x;h_1) &= u_x(x+h_1) + u_x(x-h_1)\\ &\leq H(u_x(x+h_2) + u_x(x-h_2)) = H\Delta^2 u(x;h_2).\end{aligned}$$

Conversely, let us assume (2.5). We shall prove that there is $H < \infty$ such that

$$\max_{|x-z|=r} u_{z,p}(x) \leq H \min_{|x-z|=r} u_{z,p}(x) \tag{2.6}$$

whenever $z \in \mathcal{H}$, $p \in \partial u(z)$, and $r > 0$. It is clear that (2.6) holds for u if and only if it holds for restrictions of u to 2-dimensional affine planes. Thus it suffices to prove (2.6) in the case $\mathcal{H} = \mathbb{R}^2$. It will be convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$ for this purpose. Without loss of generality we may assume that $z = 0$, $p = 0$, and $r = 4$. Let $v = u_{0,0}$ and

$$m = \min_{|x-z|=r} v(x) = \min_{-\pi \leq \theta \leq \pi} v(4e^{i\theta}).$$

We may assume $v(4) = m$. Since v satisfies (2.5) and

$$\Delta^2 v(2;2) = v(0) - 2v(2) + v(4) \leq m,$$

it follows that

$$\Delta^2 v(2; 2e^{i\theta}) \leq Lm, \quad -\pi \leq \theta \leq \pi.$$

Using this together with the obvious inequality $v(2) \leq m/2$, we obtain

$$v(2 + 2e^{i\theta}) \leq (L+1)m, \quad -\pi \leq \theta \leq \pi. \tag{2.7}$$

Setting $\theta = \pm\pi/6$ in (2.7) yields $v(3 \pm i\sqrt{3}) \leq (L+1)m$, hence

$$\Delta^2 v(3; i\sqrt{3}) \leq 2(L+1)m.$$

This and (2.5) imply that for all $\theta \in [-\pi, \pi]$

$$v(3 + \sqrt{3}e^{i\theta}) \leq 2L(L+1)m + 2v(3) \leq 2L(L+1)m + 3m/2 =: Cm.$$

By convexity of v we have $v \le Cm$ in the closed disk $\overline{B}(3,\sqrt{3})$. This disk intersects the circle $\partial B(0,4)$ along an arc $\{4e^{i\theta} : |\theta| \le \theta_0\}$, where θ_0 is a numerical constant. Therefore,

$$v(4e^{i\theta}) \le Cm, \quad |\theta| \le \theta_0. \tag{2.8}$$

The estimate (2.8) can be iterated to obtain an upper bound for $v(4e^{i\theta})$ for all $\theta \in [-\pi,\pi]$. More precisely, $v(4e^{i\theta}) \le C^N m$, where $N = \lceil \pi/\theta_0 \rceil$ is the number of required iterations. This proves (2.6) with $H = C^N$. Note that H depends only on L.

Returning to a general Hilbert space $\mathcal{H}$, we consider a section $S_u(z,p,t)$. Suppose for the moment that $S_u(z,p,t)$ is bounded, and let $R = \sup\{|x-z| : x \in S_u(z,p,t)\}$. Pick a point $x \in \partial B(z,R/2) \cap S_u(z,p,t)$ and let $y = z + H^{-1}(x-z)$. Since $u_{z,p}$ is convex,

$$u_{z,p}(y) \le H^{-1} u_{z,p}(x) + (1-H^{-1}) u_{z,p}(z) = H^{-1} t.$$

By (2.6) we have $u_{z,p}(w) \le t$ whenever $|w-z| = R/(2H)$. Therefore,

$$B(z, R/(2H)) \subset S_u(z,p,t) \subset B(z,R),$$

as required. Finally, if $S_u(z,p,t)$ is unbounded, then the preceding argument works for any R and yields $S_u(z,p,t) = \mathcal{H}$. However, the latter is impossible since u is not affine. □

Theorem 2.3 can be given a more transparent geometric interpretation. Let $ABCD$ be a rectangle in $\mathcal{H}$, and let E be its center. Assumption (2.5) says that the function $\tilde{u} = u - u(E)$ satisfies

$$L^{-1} \le \frac{\tilde{u}(A) + \tilde{u}(C)}{\tilde{u}(B) + \tilde{u}(D)} \le L.$$

In other words, the sums of values of $\tilde{u}$ along each diagonal of a rectangle are comparable to each other with a constant independent of the rectangle.

3. Monotone mappings in Hilbert spaces

The main result of this section is a surjectivity theorem for certain nonlinear operators acting on measures defined on a Hilbert space. Different versions of this result were used in [**17**] (in Euclidean spaces) and [**15**] (in Banach spaces) to construct quasiconformal and quasisymmetric mappings with prescribed properties. In the context of Hilbert spaces Theorem 3.2 is more general than the surjectivity results in [**15, 17**], although its proof is much shorter. The added generality will be used to prove Theorem 4.3 in the next section.

In this section $\mathcal{H}$ is a separable real Hilbert space, and $\mathcal{I}$ is the interval $[0,1]$ equipped with the Lebesgue measure. Let $L^2(\mathcal{I};\mathcal{H})$ be the Lebesgue-Bochner space of square integrable functions from $\mathcal{I}$ into $\mathcal{H}$. A mapping F from a Hilbert space $\mathcal{H}$ into itself is called *monotone* if

$$\langle F(x) - F(y), x - y \rangle \ge 0, \quad x, y \in \mathcal{H},$$

and *strongly monotone* if there is $c > 0$ such that

$$\langle F(x) - F(y), x - y \rangle \ge c|x-y|^2, \quad x, y \in \mathcal{H}. \tag{3.1}$$

See [**7**]. Observe that F is monotone if and only if the angle formed by the vectors $F(x) - F(y)$ and $x - y$ is at most $\pi/2$. A stronger version of this condition requires the angle to be bounded by a constant less than $\pi/2$. This concept goes back at least to Sobolevskii's paper [**23**].

DEFINITION 3.1. A mapping F from a Hilbert space $\mathcal{H}$ into itself is δ-*monotone* if there exists $\delta > 0$ such that for all $x, y \in \mathcal{H}$

$$\langle F(x) - F(y), x - y\rangle \geq \delta |F(x) - F(y)||x - y|. \tag{3.2}$$

Neither of the conditions (3.1) and (3.2) implies the other one. Unlike (3.1), inequality (3.2) may hold for a homogeneous mapping with a degree of homogeneity other than 1. This will be crucial in the next section. Let us say that $F : \mathcal{H} \to \mathcal{H}$ is *odd* if $F(-x) = -F(x)$ for all $x \in \mathcal{H}$. Note that if F is δ-monotone, then $\widetilde{F}(x) := F(x) - F(-x)$ is an odd δ-monotone mapping. In what follows I is the identity operator on $\mathcal{H}$, convolution is denoted by $*$, and the pushforward of a measure μ under F is denoted by $F_{\#}\mu$.

THEOREM 3.2. *Let $\mathcal{H}$ be a separable Hilbert space. Suppose that $F : \mathcal{H} \to \mathcal{H}$ is odd, nonconstant, uniformly continuous, and δ-monotone. Let ν be a positive measure on $\mathcal{H}$ with finite second moment, i.e., $\int_{\mathcal{H}}(1 + |x|^2)\, d\nu(x) < \infty$. Then there exists a measure μ, also with finite second moment, such that $(I + F * \mu)_{\#}\mu = \nu$.*

Given $g \in L^2(\mathcal{I}; \mathcal{H})$ and F as above, define $T_F^g : \mathcal{H} \to \mathcal{H}$ by

$$T_F^g(x) = x + \int_0^1 F(x - g(\zeta))d\zeta, \quad x \in \mathcal{H}.$$

Let us write μ_g for the pushforward of the Lebesgue measure on $\mathcal{I}$ under g. In this notation $T_F^g(x) = I + F * \mu_g$. Since F is δ-monotone, so is T_F^g. Furthermore,

$$|T_F^g(x)| = O(|x|), \quad |x| \to \infty, \tag{3.3}$$

because F has at most linear growth at infinity. Let $\mathcal{D}_F g$ denote $T_F^g \circ g$, which is a measurable function from $\mathcal{I}$ into $\mathcal{H}$. By (3.3) we have $\mathcal{D}_F g \in L^2(\mathcal{I}; \mathcal{H})$.

LEMMA 3.3. *The mapping $\mathcal{D}_F : L^2(\mathcal{I}; \mathcal{H}) \to L^2(\mathcal{I}; \mathcal{H})$ is uniformly continuous and strongly monotone.*

PROOF. Let ω_F be the modulus of continuity of F, i.e., $\omega_F(\delta) = \sup\{|F(x) - F(y)| : |x-y| \leq \delta\}$ for $\delta > 0$. By Theorem 2.3 of [**16**] F is η-quasisymmetric, where $\eta(t) = C \max\{t^\alpha, t^{1/\alpha}\}$ for some $C > 0$ and $0 < \alpha < 1$. It follows that $\omega_F(\delta) \leq C\delta^\alpha$ for sufficiently small $\delta > 0$. On the other hand, $\omega_F(n) \leq n\omega_F(1)$ for any positive integer n. Combining the above, we obtain $\omega_F(\delta) \leq C(\delta + \delta^\alpha)$ for all $\delta > 0$.

Given two functions $g, h \in L^2(\mathcal{I}; \mathcal{H})$, one can estimate the difference $\mathcal{D}_F g - \mathcal{D}_F h$ at any point $\xi \in \mathcal{I}$ as follows.

$$\begin{aligned}
|\mathcal{D}_F g(\xi) - \mathcal{D}_F h(\xi)| &\leq |g(\xi) - h(\xi)| + \int_0^1 |F(g(\xi) - g(\zeta)) - F(h(\xi) - h(\zeta))| d\zeta \\
&\leq |g(\xi) - h(\xi)| + \int_0^1 \omega_F(|(g(\xi) - g(\zeta)) - (h(\xi) - h(\zeta))|) d\zeta \\
&\leq |g(\xi) - h(\xi)| + \omega_F(|g(\xi) - h(\xi)|) + \int_0^1 \omega_F(|g(\zeta) - h(\zeta)|) d\zeta \\
&\leq (C + 1)|g(\xi) - h(\xi)| + C|g(\xi) - h(\xi)|^\alpha \\
&+ C \int_0^1 (|g(\zeta) - h(\zeta)| + |g(\zeta) - h(\zeta)|^\alpha) d\zeta
\end{aligned}$$

By Hölder's inequality we have $\|\mathcal{D}_F g - \mathcal{D}_F h\|_{L^2} \to 0$ uniformly as $\|g-h\|_{L^2} \to 0$.

It remains to prove that $\mathcal{D}_F$ is strongly monotone. To this end we compute the inner product of $\mathcal{D}_F g - \mathcal{D}_F h$ with $g - h$.

$$\begin{aligned}\langle \mathcal{D}_F g - \mathcal{D}_F h, g-h\rangle_{L^2} &= \|g-h\|_{L^2}^2 \\ &\quad + \int_0^1 \int_0^1 \langle F(g(\xi)-g(\zeta)) - F(h(\xi)-h(\zeta)), g(\xi)-h(\xi)\rangle \, d\zeta \, d\xi\end{aligned}$$

Relabeling ζ and ξ and using the assumption $F(-x) = -F(x)$, we obtain

$$\begin{aligned}&\int_0^1 \int_0^1 \langle F(g(\xi)-g(\zeta)) - F(h(\xi)-h(\zeta)), g(\xi)-h(\xi)\rangle \, d\zeta \, d\xi \\ &\quad = \int_0^1 \int_0^1 \langle F(g(\xi)-g(\zeta)) - F(h(\xi)-h(\zeta)), h(\zeta)-g(\zeta)\rangle \, d\zeta \, d\xi.\end{aligned}$$

Therefore, the double integral can be written as

$$\frac{1}{2}\int_0^1 \int_0^1 \langle F(g(\xi)-g(\zeta)) - F(h(\xi)-h(\zeta)), g(\xi)-g(\zeta)-(h(\xi)-h(\zeta))\rangle \, d\zeta \, d\xi.$$

The integrand is nonnegative because F is monotone. Therefore,

$$\langle \mathcal{D}_F g - \mathcal{D}_F h, g-h\rangle_{L^2} \geq \|g-h\|_{L^2}^2$$

as required. □

Proof of Theorem 3.2. Since every uncountable complete separable metric space is Borel isomorphic to $\mathcal{I}$ [**22**, 15.4], one can find $g \in L^2(\mathcal{I};\mathcal{H})$ such that $\nu = \mu_g$. Being a continuous strongly monotone mapping, $\mathcal{D}_F$ is surjective [**7**], hence there exists $h \in L^2(\mathcal{I};\mathcal{H})$ such that $\mathcal{D}_F h = g$. The latter implies $(I + F * \mu_h)_{\#}\mu_h = \nu$, as desired. □

The assumption of uniform continuity in Theorem 3.2 can be replaced by the homogeneity of F.

Lemma 3.4. *Let $F : \mathcal{H} \to \mathcal{H}$ be an odd homogeneous η-quasisymmetric mapping with degree of homogeneity $\alpha \in (0,1]$. Then F is uniformly continuous in $\mathcal{H}$.*

Proof. Since quasisymmetric mappings send bounded sets into bounded sets, the supremum $M := \sup\{|F(x)| : |x| \leq 1\}$ is finite. The homogeneity of F implies that $|F(x) - F(y)| \leq M|x-y|^\alpha$ whenever y is a positive multiple of x. Next, consider an arbitrary pair of distinct points $x, y \in \mathcal{H}$. Let x' be a positive multiple of x such that $|x - x'| = |x - y|$. Since

$$|F(x) - F(y)| \leq \eta(1)|F(x) - F(x')| \leq \eta(1)M|x-y|^\alpha,$$

F is Hölder continuous in $\mathcal{H}$. □

The simplest example of F that verifies the assumptions of Lemma 3.4 is $F(x) = |x|^{\alpha-1}x$, where $0 < \alpha \leq 1$. This was used in [**17**] and [**16**]. Another example will appear in the next section.

4. Quasiconformal gradients

This section is mostly concerned with mappings in the plane $\mathbb{R}^2$, which is frequently identified with $\mathbb{C}$. A set $E \subset \mathbb{C}$ is a *quasicircle* if it is the image of a circle under a quasiconformal automorphism of $\mathbb{C}$. Although quasicircles come in an abundance of shapes and sizes, a surprising theorem of Rohde [**21**] completely describes them up to a bi-Lipschitz automorphism of $\mathbb{C}$. Namely, there is an explicitly described family of "generalized snowflakes" $\{S_k\}$ such that every quasicircle coincides with $\varphi(S_k)$ for some k and some bi-Lipschitz mapping $\varphi : \mathbb{C} \to \mathbb{C}$. Inspired by this bi-Lipschitz uniformization of quasicircles, one may ask for a similar uniformization of quasiconformal mappings of $\mathbb{C}$. More precisely, one can try to factorize an arbitrary quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ as $f = \varphi \circ g$, where φ is bi-Lipschitz and g has some special structure. The key question here is what kind of structure one can expect g to have.

Some regularity problems for uniformly elliptic equations in two dimensions [**11**, Ch.12] naturally lead one to consider *quasiconformal gradients* [**1, 18**]. A quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ is called a quasiconformal gradient if $\operatorname{Im} \partial f/\partial \bar{z} = 0$ a.e. in $\mathbb{C}$. For any such f one can construct a uniformly elliptic equation with a solution u such that $\partial u/\partial z = f$ (this idea goes back to [**20**]). This motivates the following

Question 4.1. Does every quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ admit a factorization $f = \varphi \circ g$, where φ is bi-Lipschitz and g is a quasiconformal gradient?

A quasiconformal mapping f is bi-Lipschitz if and only if its Jacobian determinant $J_f := \det Df$ is pinched between two positive constants [**21**]. Therefore, Question 4.1 can be stated in a different form: given a quasiconformal mapping f, can one find a quasiconformal gradient g such that $C^{-1} J_f \le J_g \le C J_f$ for some constant C? It is natural to approach this question by studying the sets where the Jacobians of f and g assume the values 0 or ∞. Since such sets must have measure zero and the Jacobians are defined only a.e., some clarification is required here. When ψ is a real-valued function defined on a subset of $\mathbb{R}^n$, we write $\operatorname*{ess\,lim}_{y\to x} \psi(y) = a$ if there is a set $Z \subset \mathbb{R}^n$ such that $\mathcal{L}^n(Z) = 0$ and $\lim_{y\to x,\ y\notin Z} \psi(y) = a$. Here and in what follows $\mathcal{L}^n$ stands for the n-dimensional Lebesgue measure. If ψ is locally integrable, then its precise representative

$$\widetilde{\psi}(x) = \begin{cases} \lim\limits_{r\to 0} \dfrac{1}{\mathcal{L}^n(B(x,r))} \displaystyle\int_{B(x,r)} \psi(y)\, d\mathcal{L}^n(y), & \text{if the limit exists;} \\ 0 & \text{otherwise;} \end{cases}$$

agrees with ψ at a.e. point [**8**]. The following elementary result clarifies what we mean by saying that J_f attains the values 0 or ∞.

Proposition 4.2. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is quasiconformal. For any $x \in \mathbb{R}^n$ and $\lambda \in \{0, \infty\}$ the following are equivalent:*

(i) $\operatorname*{ess\,lim}_{y\to x} J_f(y) = \lambda$;

(ii) $\lim_{y\to x} \widetilde{J_f}(y) = \lambda$;

(iii) $\lim\limits_{\substack{y,z\to x\\ y\ne z}} \dfrac{|f(y)-f(z)|}{|y-z|} = \lambda.$

Proof. Since f is quasiconformal, there is $C > 1$ such that

$$C^{-1}|f(y) - f(z)|^n \le \int_{B(y,|y-z|)} J_f(w)\, d\mathcal{L}^n(w) \le C|f(y) - f(z)|^n$$

for any $y, z \in \mathbb{R}^n$. This proves the chain of implications (i)⇒(iii)⇒(ii). The implication (ii)⇒(i) holds because $\widetilde{J_f} = J_f$ a.e. □

For any set $E \subset \mathbb{C}$ of Hausdorff dimension $\dim E < 1$ one can find a quasiconformal mapping whose Jacobian has essential limit 0 (or ∞) at every point of E [**17**]. If there were no quasiconformal *gradients* with this property, Question 4.1 would have a negative answer. However, we prove the following

Theorem 4.3. *For every set $E \subset \mathbb{C}$ of Hausdorff dimension less than* 1 *and for any $\lambda \in \{0, \infty\}$ there exists a quasiconformal gradient $g : \mathbb{C} \to \mathbb{C}$ such that* $\operatorname{ess\,lim}_{y\to x} J_g(y) = \lambda$ *for all $x \in E$.*

We require the following result from [**16**].

Proposition 4.4. *For any $\alpha \in (0, 1)$ there exist $k \in (0, 1)$ and an odd δ-monotone mapping $g_\alpha : \mathbb{C} \to \mathbb{C}$ such that g_α is nonconstant, homogeneous of degree α, and*

$$\operatorname{Im} \frac{\partial g_\alpha}{\partial \bar{z}} = 0 \quad \text{a.e. in } \mathbb{C}. \tag{4.1}$$

We claim that the mapping g_α of Proposition 4.4 satisfies

$$\frac{|g_\alpha(z) - g_\alpha(\zeta)|}{|z - \zeta|} \ge c \max\{|z|, |\zeta|\}^{\alpha-1}, \quad z \ne \zeta, \tag{4.2}$$

where $c > 0$ does not depend on z and ζ. Indeed, let

$$c := \inf \left\{ \frac{|g_\alpha(z) - g_\alpha(\zeta)|}{|z - \zeta|} : |z| \le 1,\ |\zeta| \le 1,\ z \ne \zeta \right\}. \tag{4.3}$$

Inequality (4.2) will follow once we prove $c > 0$. Consider z and ζ as in (4.3). Since the quotient involved in (4.3) is homogeneous of degree $\alpha - 1 < 0$, we may assume without loss of generality that $|\zeta| = 1$. Let ζ' be a positive multiple of ζ such that $|\zeta - \zeta'| = |z - \zeta|$. The quasiconformality of g_α implies

$$|g_\alpha(z) - g_\alpha(\zeta)| \ge C|g_\alpha(\zeta') - g_\alpha(\zeta)|$$

where $C > 0$ depends only on g_α. Let $\lambda = |\zeta'|$. Clearly $0 < \lambda \le 3$. The homogeneity of g_α implies

$$|g_\alpha(\zeta') - g_\alpha(\zeta)| = |\lambda^\alpha - 1||g_\alpha(\zeta)|.$$

Since $|g_\alpha(\zeta)|$ is bounded from below on the unit circle, it remains to observe that the ratio

$$\frac{|\lambda^\alpha - 1|}{|z - \zeta|} = \frac{|\lambda^\alpha - 1|}{|\lambda - 1|}$$

is bounded from below when $0 < \lambda \le 3$. Having proved (4.2), we can now proceed to

Proof of Theorem 4.3. Let $E \subset \mathbb{R}^n$ be a set with $\dim E < 1$. Classical results of the potential theory [**19**] guarantee that there exist $\alpha \in (0, 1)$ and a measure μ with finite second moment such that

$$\int_{\mathbb{C}} |z - w|^{\alpha-1}\, d\mu(w) = \infty \quad \text{for all } z \in E. \tag{4.4}$$

See [**17**] for a detailed discussion. The convolution $G_\mu := g_\alpha * \mu$ is a δ-monotone, hence quasiconformal, mapping. Moreover, it is a quasiconformal gradient by virtue of (4.1). Given two distinct points $\zeta, \xi \in \mathbb{C}$, we use (4.2) and the δ-monotonicity of g_α to obtain

$$\begin{aligned}\frac{|G_\mu(\zeta) - G_\mu(\xi)|}{|\zeta - \xi|} &\geq \operatorname{Re} \frac{G_\mu(\zeta) - G_\mu(\xi)}{\zeta - \xi} \\ &= \int_{\mathbb{C}} \operatorname{Re} \frac{g_\alpha(\zeta - w) - g_\alpha(\xi - w)}{\zeta - \xi} d\mu(w) \\ &\geq \delta \int_{\mathbb{C}} \frac{|g_\alpha(\zeta - w) - g_\alpha(\xi - w)|}{|\zeta - \xi|} d\mu(w) \\ &\geq c\delta \int_{\mathbb{C}} \max\{|\zeta - w|, |\xi - w|\}^{\alpha - 1} d\mu(w).\end{aligned}$$

This together with (4.4) and the lower semicontinuity of Riesz potentials imply that for any $z \in E$

$$\frac{|G_\mu(\zeta) - G_\mu(\xi)|}{|\zeta - \xi|} \to \infty, \quad \text{as } \zeta, \xi \to z.$$

By Proposition 4.2 this completes the proof of the case $\lambda = \infty$.

Naturally, we want to use a mapping of the form G_μ^{-1} to prove Theorem 4.3 for $\lambda = 0$. This is possible because the inverse of a quasiconformal gradient is itself a quasiconformal gradient:

$$\frac{\partial G^{-1}}{\partial \bar{z}} \circ G = -J_G^{-1} \frac{\partial G}{\partial \bar{z}}$$

is real-valued if $\partial G / \partial \bar{z}$ is. To find an appropriate μ, we apply Theorem 3.2 with $F = g_\alpha$ and ν chosen so that

$$\int_{\mathbb{C}} |z - w|^{\alpha - 1} \, d\nu(w) = \infty \quad \text{for all } z \in E. \tag{4.5}$$

Let $G^\mu = (I + G_\mu)^{-1}$, where I is the identity map on $\mathbb{C}$. Since G_μ is a monotone mapping, G^μ is a contraction. Therefore, for every $z \in E$ we have

$$\begin{aligned}\int_{\mathbb{C}} |G^\mu(z) - \zeta|^{\alpha - 1} \, d\mu(\zeta) &= \int_{\mathbb{C}} |G^\mu(z) - G^\mu(w)|^{\alpha - 1} \, d\nu(w) \\ &\geq \int_{\mathbb{C}} |z - w|^{\alpha - 1} \, d\nu(w) = \infty.\end{aligned}$$

As in the case $\lambda = \infty$, we obtain that the Jacobian of $I + G_\mu$ is infinite at every point of $G^\mu(E)$. Thus, the Jacobian of G^μ vanishes on E. □

Although the Jacobian of a quasiconformal mapping cannot vanish on any rectifiable curve, it can be infinite along a line segment. Examples of this kind can be found in [**24**] and [**14**]. At present we do not know if there is a quasiconformal gradient g such that J_g is infinite on a line segment. If no such g exists, then the answer to Question 4.1 is negative.

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