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INTERIOR RADII OF SYMMETRIC NOT LEANING DOMAINS

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Many extremal problems for classes of analytic functions are reduced to problems on not leaning domains (see, e. g., [1], pp. 552–554; [2]; survey of new results in this direction can be found in [3]). Let D be a domain of the complex sphere $\overline{\mathbb{C}}$. We denote by $g_D(z, z_0)$ Green's function of the domain D with the pole z_0; the interior radius of D with respect to z_0 is

$$r(D, z_0) \overset{\text{def}}{=} \begin{cases} \exp\left[\lim\limits_{z\to z_0}(g_D(z,z_0) + \log|z - z_0|)\right], & z_0 \neq \infty; \\ \exp\left[\lim\limits_{z\to z_0}(g_D(z,z_0) - \log|z|)\right], & z_0 = \infty. \end{cases}$$

Set $D^* \overset{\text{def}}{=} \{z : 1/\overline{z} \in D\}$.

The article is devoted to solving the following problem, posed in survey [4]. Let B_k, $k = 0, \dots, n$, be pairwise not leaning domains in $\overline{\mathbb{C}}$, $a_k \in B_k$. Find the least upper bound of the product $\prod\limits_{k=0}^{n} r(B_k, a_k)$ under the conditions $a_0 = 0$, $|a_k| = 1$, $B_k = B_k^*$, $k = 1, \dots, n$. This statement strengthens Bakhtina's problem (see [5]), where the simple connection of domains B_k was supposed and the condition $B_0 \subset \{z : |z| < 1\}$ was fulfilled; in these assumptions, in [5] the qualitative characteristic of the extremal configuration in terms of quadratic differentials was obtained (see [6], p. 48).

Theorem. *Let $B_0, \dots, B_n$ ($n > 2$) be not leaning domains in $\overline{\mathbb{C}}$; $a_k \in B_k$, $k = 0, \dots, n$; $a_0 = 0$, $|a_k| = 1$, $k = 1, \dots, n$; $B_k = B_k^*$, $k = 1, \dots, n$. Then*

$$\prod_{k=0}^{n} r(B_k, a_k) \le \frac{2^{2n+1/n}}{(n^2-2)^{n/2+1/n}} \left(\frac{n-\sqrt{2}}{n+\sqrt{2}}\right)^{\sqrt{2}}. \tag{1}$$

If, in addition, the domains B_k possess classical Green's functions, then the equality in (1) is attained if and only if the points a_k and domains B_k are poles and circular domains, respectively, of the quadratic differential

$$Q(z)dz^2 = -\frac{(\alpha z)^{2n} + (2n^2-2)(\alpha z)^n + 1}{z^2((\alpha z)^n - 1)^2} dz^2, \quad |\alpha| = 1.$$

Sketch of the proof. Let $a_k = \exp(i\theta_k)$, $0 = \theta_1 < \dots < \theta_n < \theta_{n+1} = 2\pi$, $\varphi_k = \theta_{k+1} - \theta_k$, $k = 1, \dots, n$. Consider the two cases.

1. Suppose that $\varphi_k \le \pi\sqrt{2}$, $k = 1, \dots, n$. Let $D_i = \{z : |z| < 1,\ \theta_i < \arg z < \theta_{i+1}\}$. Let us apply a separating transformation (see [7]) of each domain B_k with respect to an appropriate family of functions which conformally map the domains D_i, D_i^*, $i = 1, \dots, n$, onto a halfplane. By the same token the determination of the least upper bound of the left side of (1) is reduced to the estimation

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of n products of the form $r^a(G_1, \infty)r(G_2, 2i)r(G_3, -2i)$, where G_1, G_2, and G_3 are not leaning domains in $\overline{\mathbb{C}}$. Estimating the products mentioned by means of theorem 1 in [8] and investigating the expression obtained for extremum, we arrive at inequality (1).

2. Let, for the sake of definiteness, $\varphi_n > \pi\sqrt{2}$. Repeating the proof of theorem 4 in [7] with regard for this condition, we get for the product $\prod_{k=0}^{n} r(B_k, a_k)$ the upper estimate, which is strictly lesser than the right side in (1).

Assertion about the case of equality can be readily derived from the corresponding assertion of theorem 5 in [7].

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