Math 602 Final Exam (05/11/11). Max total score 50.

Your name: ____

Out of the **seven** problems, do any **five** (worth 10 points each). You may use your textbook and notes, but you should work individually.
Due at 4:30pm Wednesday, May 11.

1. Define $f\colon [0,1] \to R$ as follows: $f(0) = 0$, and for $x \neq 0$ set $f(x) = (-1)^{\lfloor 1/x \rfloor}$ where $\lfloor 1/x \rfloor$ is the greatest integer than does not exceed $1/x$.

Prove that $f \in \mathscr{R}$, that is, f is integrable with respect to dx.

2. Let $E \subset \mathbb{R}^k$ be a set such that for any $p, q \in E$ there exists a rectifiable curve $\gamma\colon [a, b] \to E$ such that $\gamma(a) = p$ and $\gamma(b) = q$. (Here $[a, b]$ can be any closed interval on the real line.) Define

$$d(p, q) = \inf\{\Lambda(\gamma)\colon \gamma \text{ is as above}\},$$

where $\Lambda(\gamma)$ is the length of γ. Prove that d is a metric on the set E.

3. Give an example of a sequence of continuous functions $f_n\colon [0,1] \to [0,\infty)$ which has the following three properties: (i) $f_n \to 0$ pointwise; (ii) $\displaystyle\int_0^1 f_n(x)\,dx \to \infty$; (iii) $\displaystyle\int_0^1 x f_n(x)\,dx \to 0$.

4. Suppose that $f\colon [-1,1] \to \mathbb{R}$ is a differentiable function. Prove that for every $\epsilon > 0$ there exists a polynomial P such that $|f(x) - P(x)| \leqslant \epsilon|x|$ for all $x \in [-1,1]$.

5. For $x \in \mathbb{R}$ define $f(x) = \sum_{n=1}^{\infty} n^{-3} \sin nx$. Prove that there exists a constant L such that $|f(x) - f(y)| \leqslant L|x - y|$ for all $x, y \in \mathbb{R}$.

6. Let $h\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $h(0) = 0$. Define the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \frac{xy^2 h(x+y)}{x^2 + y^2} \quad \text{if } (x, y) \neq (0, 0); \quad f(0, 0) = 0.$$

Prove that f is differentiable at $(0, 0)$.

7. Let $h\colon \mathbb{R} \to (0,\infty)$ be a continuously differentiable function. Define the mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ as follows: $f(x,y) = (u,v)$ where

$$u = e^{2x} h(y) \sin y; \qquad v = e^{2x} h(y) \cos y$$

Prove that the image of any open set $U \subset \mathbb{R}^2$ under f is also an open set in $\mathbb{R}^2$.