Math 602 Exam 3 (04/05/11). Solutions.

1. Suppose that $f\colon [0,1] \to \mathbb{R}$ and $g\colon [0,1] \to \mathbb{R}$ are continuous functions such that $f(x) < g(x)$ for all $x \in [0,1]$. Prove that there exists a polynomial P such that $f(x) < P(x) < g(x)$ for all $x \in [0,1]$.

Proof. Since $g - f$ is a positive continuous function on compact set $[0,1]$, its infimum, denoted by ϵ, is positive. Let $h = \dfrac{f+g}{2}$: this is also a continuous function. By the Weierstrass approximation theorem, there exists a polynomial P with real coefficients such that $|P(x) - h(x)| < \dfrac{\epsilon}{2}$ for all $x \in [0,1]$. It follows from the triangle inequality that

$$P(x) < h(x) + \frac{\epsilon}{2} \leqslant \frac{f(x)+g(x)}{2} + \frac{g(x)-f(x)}{2} = g(x)$$
$$P(x) > h(x) - \frac{\epsilon}{2} \geqslant \frac{f(x)+g(x)}{2} - \frac{g(x)-f(x)}{2} = f(x)$$

2. Let $f(x) = 1$ if $x \in [0,\pi]$ and $f(x) = 0$ if $x \in [-\pi, 0]$. Compute the Fourier series of f and show that its sum $\lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{inx}$ is equal to $1/2$ at $x = 0$.

Proof. For $n \neq 0$,

$$c_n = \frac{1}{2\pi}\int_0^{\pi} e^{-inx}\,dx = \frac{1}{2\pi}\frac{e^{-in\pi}-1}{-in} = \frac{i}{2\pi n}((-1)^n - 1)$$

which is $-i/(\pi n)$ when n is odd and 0 when n is even. For $n = 0$ we get $c_0 = \dfrac{1}{2\pi}\displaystyle\int_0^{\pi} 1\,dx = 1/2$.
At $x = 0$, the partial sums of the Fourier series are

$$\frac{1}{2} + \sum_{n=-N}^{N} c_n = \frac{1}{2}$$

because $c_{-n} = -c_n$ for all $n \neq 0$. Hence the limit as $N \to \infty$ is also $1/2$.

3. Suppose that $f\colon [-\pi,\pi] \to \mathbb{R}$ is a differentiable function such that f' is continuous on $x \in [-\pi,\pi]$. Let c_n, $n \in \mathbb{Z}$, be the Fourier coefficients of f. Prove that there exists a constant M such that $|c_n| \leqslant M/|n|$ for all $n \in \mathbb{Z} \setminus \{0\}$.

Proof. Evaluate c_n using integration by parts:

$$2\pi c_n = \int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = f(\pi)\frac{e^{-in\pi}}{-in} - f(-\pi)\frac{e^{in\pi}}{-in} - \int_{-\pi}^{\pi} f'(x)\frac{e^{-inx}}{-in}\,dx$$
$$= \frac{f(\pi) - f(-\pi)}{-in} + \frac{1}{in}\int_{-\pi}^{\pi} f'(x)\frac{e^{-inx}}{}dx$$

By the triangle inequality,

$$2\pi|c_n| \leqslant \frac{2}{n}\sup|f| + \frac{2\pi}{n}\sup|f'|.$$

We can take $M = \frac{1}{\pi}\sup|f| + \sup|f'|$.
Remark. If $f(\pi) = f(-\pi)$, then the result can be improved to $nc_n \to 0$ by the Riemann-Lebesgue lemma applied to f'.

4. Consider the power series $\sum_{n=0}^{\infty} c_n x^n$ in which the coefficients $c_n \in \mathbb{R}$ satisfy $c_{n+2} = c_n$ for all n. Prove that:

(a) The series converges for $x \in (-1, 1)$.

(b) Its sum is a rational function of x; that is, the ratio of two polynomials.

Proof. (a) The sequence $\{c_n\}$ is bounded by $M = \max\{|c_0|, |c_1|\}$. For $x \in (-1, 1)$ the series $\sum c_n x^n$ converges by comparison to $\sum M|x|^n$, where the latter series is geometric with ratio $|x| < 1$.

(b) Consider a partial sum of the series by $(1 - x^2)$ which runs up to $2N$ (just for convenience):

$$\sum_{n=0}^{2N} c_n x^n = c_0 \sum_{k=0}^{N} x^{2k} + c_1 \sum_{k=1}^{N} x^{2k-1} = c_0 \frac{1 - x^{2N+2}}{1 - x^2} + c_1 \frac{x - x^{2N+1}}{1 - x^2}$$

The limit as $N \to \infty$ is $\dfrac{c_0}{1 - x^2} + \dfrac{c_1 x}{1 - x^2} = \dfrac{c_0 + c_1 x}{1 - x^2}$. Since this limit is equal to the sum of the series, $f(x)$ is a rational function.

5. For $x \in \mathbb{R}$ define $f(x) = \displaystyle\sum_{n=1}^{\infty} 2^{-n} \cos nx$. Prove that the integral $\displaystyle\int_{-\pi}^{\pi} f(x)^2\, dx$ exists and find its value.

Proof. The series that defines f converges uniformly by the Weierstrass test: $\sum 2^{-n} < \infty$. Therefore, the integral exists and is equal to the limit of integrals of f_N^2 where f_N is a partial sum of the series. Using the identity $\cos nx = \dfrac{e^{inx} + e^{-inx}}{2}$, rearrange f_N as

$$f_N(x) = \frac{1}{2} \sum_{0<|n|\leqslant N} 2^{-|n|} e^{inx}$$

Hence

$$f_N(x)^2 = \frac{1}{4} \sum_{0<|n|,|m|\leqslant N} 2^{-|n|-|m|} e^{i(n+m)x}$$

The terms with $n + m \neq 0$ integrate to zero over $[-\pi, \pi]$, since e^{ikx} has antiderivative $(ik)^{-1} e^{ikx}$, which is 2π-periodic. The remaining terms are identically equal to $2^{-2|n|}$.

$$\int_{-\pi}^{\pi} f_N(x)^2\, dx = \frac{1}{4} \sum_{0<|n|\leqslant N} \int_{-\pi}^{\pi} 2^{-2|n|}\, dx = \frac{\pi}{2} \sum_{0<|n|\leqslant N} 2^{-2|n|} = \pi \sum_{n=1}^{N} 2^{-2n}.$$

Let $N \to \infty$ and compute the sum of the geometric series:

$$\int_{-\pi}^{\pi} f(x)^2\, dx = \pi \sum_{n=1}^{\infty} 2^{-2n} = \pi \frac{1/4}{1 - 1/4} = \frac{\pi}{3}$$

6. “The one-sided limit $\lim\limits_{x\to 0+} x^p \log x$ exists for every $p > 0$.” True: Apply L’H to $\dfrac{\log x}{x^{-p}}$.

7. “If P is a polynomial of degree $d \geqslant 2$ with complex coefficients, then there exists $z \in \mathbb{C}$ such that $P(z) = z$.” True: apply FTA to $P(z) - z$.