Math 602 Exam 3 (04/05/11). Max total score 50.

Your name: ____

READ THIS FIRST: Do not open the exam booklet until told to do so. Out of the first **five** problems, do any **four** (worth 10 points each). If you attempt all five problems, indicate which one is not to be graded. The exam concludes with two True/False questions worth 5 points each. You may not use the textbook or notes.

Part I: Do four out of five problems. If you attempt all five problems, indicate which one is not to be graded. Support your claims.

1. Suppose that $f\colon [0,1] \to \mathbb{R}$ and $g\colon [0,1] \to \mathbb{R}$ are continuous functions such that $f(x) < g(x)$ for all $x \in [0,1]$. Prove that there exists a polynomial P such that $f(x) < P(x) < g(x)$ for all $x \in [0,1]$.

2. Let $f(x) = 1$ if $x \in [0, \pi]$ and $f(x) = 0$ if $x \in [-\pi, 0)$. Compute the Fourier series of f and use it to show that $\lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{inx}$ is equal to $1/2$ at $x = 0$.

3. Suppose that $f\colon [-\pi, \pi] \to \mathbb{R}$ is a differentiable function such that f' is continuous on $x \in [-\pi, \pi]$. Let c_n, $n \in \mathbb{Z}$, be the Fourier coefficients of f. Prove that there exists a constant M such that $|c_n| \leqslant M/|n|$ for all $n \in \mathbb{Z} \setminus \{0\}$.

4. Consider the power series $\sum_{n=0}^{\infty} c_n x^n$ in which the coefficients $c_n \in \mathbb{R}$ satisfy $c_{n+2} = c_n$ for all n.
Prove that: (a) The series converges for $x \in (-1, 1)$.
(b) Its sum is a rational function of x; that is, the ratio of two polynomials.

5. For $x \in \mathbb{R}$ define $f(x) = \sum_{n=1}^{\infty} 2^{-n} \cos nx$. Prove that the integral $\int_{-\pi}^{\pi} f(x)^2 \, dx$ exists and find its value.

Part II: True/False questions, 5 points each. You do not need to support your claims in this part.

6. "The one-sided limit $\lim_{x \to 0+} x^p \log x$ exists for every $p > 0$."

True ______ *False* ______

7. "If P is a polynomial of degree $d \geqslant 2$ with complex coefficients, then there exists $z \in \mathbb{C}$ such that $P(z) = z$."

True ______ *False* ______