MATH 602 EXAM 2 SOLUTIONS.

1. Let $\mathscr{F}$ be an equicontinuous family of functions from $\mathbb{R}$ to $\mathbb{R}$.
Prove that the family $\mathscr{F}_1 = \{f \circ g \colon f, g \in \mathscr{F}\}$ is also equicontinuous.

Proof. Fix $\epsilon > 0$. By the equicontinuity of $\mathscr{F}$, there exists $\delta > 0$ such that for all $f \in \mathscr{F}$ we have $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Using the equicontinuity again, find $\delta' > 0$ such that for all $f \in \mathscr{F}$ we have $|f(x) - f(y)| < \delta$ whenever $|x - y| < \delta'$.
Now let f and g be any functions from $\mathscr{F}$, and suppose $|x - y| < \delta'$, $x, y \in \mathbb{R}$. From the above we have $|f(g(x)) - f(g(y))| < \epsilon$, as required. □

2. Give an example of a sequence of continuous functions $f_n \colon [0,1] \to \mathbb{R}$ which has the following three properties: (i) $f_n \to 0$ pointwise; (ii) $\displaystyle\int_0^1 |f_n|\,dx \to 0$; (iii) $\displaystyle\int_0^1 f_n^2\,dx \to \infty$.

Proof. Recall an example given in class to demonstrate that pointwise convergence does not imply convergence of integrals: the sequence

$$g_n = \max(0, n - n^2|x - 1/n|) = \begin{cases} n^2 x, & 0 \leqslant x \leqslant 1/n \\ n^2(2/n - x), & 1/n \leqslant x \leqslant 2/n \\ 0, & 2/n \leqslant x \leqslant 1 \end{cases}$$

converges to 0 pointwise but $\int_0^1 g\,dx = 1$. Also,

$$\int_0^1 g_n^2\,dx = 2\int_0^{1/n} n^4 x^2\,dx = \frac{2}{3}n \to \infty$$

Thus, we already have (i) and (iii), but need to make the function a bit smaller to also achieve (ii).
One can try guess-and-check, but the more reliable method is to introduce a coefficient $c_n > 0$ to be chosen later: $f_n(x) = c_n g_n(x)$. No matter what c_n is, (i) holds because $f_n(0) = 0$ and for any $x \in (0,1]$ we have $f_n(x) = 0$ when $n > 2/x$. Since

$$\int_0^1 |f_n|\,dx = c_n \quad \text{and} \quad \int_0^1 f_n^2\,dx = c_n^2 \cdot \frac{2}{3}n$$

We need $c_n \to 0$ and $nc_n^2 \to \infty$; both are achieved with $c_n = n^{-1/3}$. Thus, $f_n = n^{-1/3} g_n$ satisfies (i), (ii), and (iii). □

3. Prove that $\displaystyle\int_0^1 \frac{1}{1+x}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$.

Proof. The geometric series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1,$$

converges uniformly on $[0,b]$ for any $0 < b < 1$, but it does not converge uniformly on $[0,1)$. There are two ways to get around this difficulty: (A) work with $[0,b]$ and then let $b \to 1$; (B) modify the series to achieve uniform convergence on $[0,1]$.

(A) Let $0 < b < 1$. Applying the Weierstrass text to $\sum_{n=0}^{\infty}(-1)^n x^n$ on $[0, b]$, we find that convergence is uniform because $\sum_{n=0}^{\infty} b^n < \infty$. Thus, the integral of the sum is the sum of integrals:

$$\int_0^b \frac{1}{1+x}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} b^n, \qquad 0 < b < 1. \tag{1}$$

As $b \to 1$, the left side of (1) tends to $\int_0^1 \frac{1}{1+x}\,dx$ because

$$\left| \int_0^1 \frac{1}{1+x}\,dx - \int_0^b \frac{1}{1+x}\,dx \right| = \int_b^1 \frac{1}{1+x}\,dx \leqslant (1-b) \to 0.$$

It remains to show that

$$\lim_{b\to 1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} b^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \tag{2}$$

To this end, consider the function

$$F(b) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} b^n, \qquad 0 \leqslant b \leqslant 1. \tag{3}$$

The series defining F is uniformly convergent on $[0, 1]$ by #3 in Homework 4. Thus F is continuous on $[0, 1]$. In particular, $\lim_{b\to 1} F(b) = F(1)$, which is exactly (2).

(B) Following the earlier example of $1/(1 + x^2)$, begin with

$$\frac{2}{1+x} = 1 + 1 - x - x + x^2 + x^2 - x^3 - x^3 + x^4 + x^4 - \dots, \qquad |x| < 1 \tag{4}$$

and group the terms as

$$\frac{2}{1+x} = 1 + \sum_{n=0}^{\infty} x^{2n}(1-x)^2, \qquad |x| < 1. \tag{5}$$

The new series converges at $x = 1$ as well, and the equality (5) holds at $x = 1$. Moreover, each term is nonnegative, which means that partial sums form an increasing series of functions, Since both the partial sums and their limit are continuous on $[0, 1]$, we conclude (by Theorem 7.13) that the convergence is uniform on $[0, 1]$. Hence we can integrate term by term:

$$\int_0^1 \frac{2}{1+x}\,dx = 1 + \sum_{n=0}^{\infty} \int_0^1 x^{2n}(1-x)^2\,dx = 1 + \sum_{n=0}^{\infty} \left(\frac{1}{2n+1} - \frac{2}{2n+2} + \frac{1}{2n+3} \right). \tag{6}$$

It remains to prove that the series in (6) has the expected sum. To this end, rearrange its partial sum:

$$1 + \sum_{n=0}^{N} \left(\frac{1}{2n+1} - \frac{2}{2n+2} + \frac{1}{2n+3} \right) = \frac{1}{2N+3} + 2\sum_{n=0}^{N} \left(\frac{1}{2n+1} - \frac{1}{2n+2} \right)$$

and notice that

$$2\sum_{n=0}^{N} \left(\frac{1}{2n+1} - \frac{1}{2n+2} \right) = 2\sum_{k=0}^{2N+1} \frac{(-1)^k}{k+1} \to 2\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$$

as $N \to \infty$. □

4. For $x \in [0,1]$, let $f_0(x) = x$ and define $f_n(x) = f_{n-1}(x) \cdot \big(1 - f_{n-1}(x)\big)$ for $n = 1, 2, \dots$. Prove that $f_n \to 0$ uniformly on $[0,1]$.

Proof. Since f_n is continuous and $f_n = f_{n-1} - f_{n-1}^2 \leqslant f_{n-1}$, it suffices to prove pointwise convergence to 0 (uniform convergence follows from Theorem 7.13). For any $x \in [0,1]$ the values $f_n(x)$ are between 0 and 1, which is readily seen by induction (the product of two numbers in $[0,1]$ lies in the same interval). For a fixed $x \in [0,1]$, the sequence $\{f_n(x)\}$ is decreasing and bounded, and therefore has a limit which we denote by L. Also, $f_{n+1}(x) = f_n(x) \cdot (1 - f_n(x)) \to L(1-L)$ as $n \to \infty$. Since $\lim_{n\to\infty} f_{n+1} = \lim_{n\to\infty} f_n$, it follows that $L = L(1-L)$. This is only possible when $L = 0$. ☐

5. Let $\mathscr{C}$ be the set of all continuous functions from $[0,1]$ to $\mathbb{R}$. Given $f, g \in \mathscr{C}$, consider the set $N(f,g) := \{x \in [0,1] \colon f(x) \neq g(x)\}$ and define

$$d(f,g) = \begin{cases} 0 & \text{if } N(f,g) = \varnothing; \\ \sup N(f,g) & \text{otherwise.} \end{cases}$$

Show that d is a metric on $\mathscr{C}$, and then prove that the metric space $(\mathscr{C}, d)$ is not complete.

Proof. Properties of metric: (i) $d \geqslant 0$ by definition. Also, $d(f,g) = 0$ if and only if $f(x) = g(x)$ for all $x > 0$, in which case the continuity implies $f(0) = g(0)$ as well.
(ii) $d(f,g) = d(g,f)$ because $N(f,g) = N(g,f)$.
(iii) To prove $d(f,h) \leqslant d(f,g) + d(g,h)$, take $x > d(f,g) + d(g,h)$ and observe that $f(x) = g(x)$ and $g(x) = h(x)$ by the definition of d. Hence $f(x) = h(x)$ for all such x, which yields the desired inequality.
To prove that $(\mathscr{C}, d)$ is not complete, we need a Cauchy sequence in $(\mathscr{C}, d)$ that fails to have a limit in the sense of $(\mathscr{C}, d)$. One way to create a Cauchy sequence is to cut off the same function at different places, for example

$$f_n(x) = \min(1/x, n), \qquad f_n(0) = n,$$

is the cutoff of $1/x$ at level n. For $m > n \geqslant N$ we have $d(f_n, f_m) = 1/n \leqslant 1/N \to 0$ as $N \to \infty$. Thus the sequence is Cauchy.
Suppose that there is $f \in \mathscr{C}$ such that $d(f, f_n) \to 0$ as $n \to \infty$. Being continuous on $[0,1]$, the function f is bounded: say, $|f| \leqslant M$ for some constant M. Fix a number $\epsilon \in (0,1)$ such that $\epsilon < 1/M$. For all $n > 1/\epsilon$ we have $f_n(\epsilon) = 1/\epsilon > M \geqslant f(\epsilon)$. Hence $d(f, f_n) \geqslant \epsilon$, a contradiction. ☐

6. "If $f_n \colon [0,1] \to \mathbb{R}$ is differentiable on $[0,1]$ for each n, and $f_n \to f$ uniformly on $[0,1]$, then f is differentiable on $[0,1]$."
False: counterexamples were constructed in Homework 4.

7. "If $f_n \colon [0,1] \to \mathbb{R}$ is Riemann integrable on $[0,1]$ for each n, and $f_n \to f$ uniformly on $[0,1]$, then f is Riemann integrable on $[0,1]$."
True, by Theorem 7.16.