Math 602 Exam 2 (03/03/11). Max total score 50.

Your name: ____

READ THIS FIRST: Do not open the exam booklet until told to do so. Out of the first **five** problems, do any **four** (worth 10 points each). If you attempt all five problems, indicate which one is not to be graded. The exam concludes with two True/False questions worth 5 points each. You may not use the textbook or notes.

Part I: Do four out of five problems. If you attempt all five problems, indicate which one is not to be graded. Support your claims.

1. Let $\mathscr{F}$ be an equicontinuous family of functions from $\mathbb{R}$ to $\mathbb{R}$.
Prove that the family $\mathscr{F}_1 = \{f \circ g \colon f, g \in \mathscr{F}\}$ is also equicontinuous.

2. Give an example of a sequence of continuous functions $f_n\colon [0,1] \to \mathbb{R}$ which has the following three properties: (i) $f_n \to 0$ pointwise; (ii) $\displaystyle\int_0^1 |f_n|\,dx \to 0$; (iii) $\displaystyle\int_0^1 f_n^2\,dx \to \infty$.

3. Prove that $\displaystyle\int_0^1 \frac{1}{1+x}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$.

4. For $x \in [0,1]$, let $f_0(x) = x$ and define $f_n(x) = f_{n-1}(x) \cdot \big(1 - f_{n-1}(x)\big)$ for $n = 1, 2, \ldots$. Prove that $f_n \to 0$ uniformly on $[0,1]$.

5. Let $\mathscr{C}$ be the set of all continuous functions from $[0,1]$ to $\mathbb{R}$. Given $f, g \in \mathscr{C}$, consider the set $N(f,g) := \{x \in [0,1] \colon f(x) \neq g(x)\}$ and define

$$d(f,g) = \begin{cases} 0 & \text{if } N(f,g) = \varnothing; \\ \sup N(f,g) & \text{otherwise.} \end{cases}$$

Show that d is a metric on $\mathscr{C}$, and then prove that the metric space $(\mathscr{C}, d)$ is not complete.

Part II: True/False questions, 5 points each. You do not need to support your claims in this part.

6. "If $f_n\colon [0,1] \to \mathbb{R}$ is differentiable on $[0,1]$ for each n, and $f_n \to f$ uniformly on $[0,1]$, then f is differentiable on $[0,1]$."

True ______ *False* ______

7. "If $f_n\colon [0,1] \to \mathbb{R}$ is Riemann integrable on $[0,1]$ for each n, and $f_n \to f$ uniformly on $[0,1]$, then f is Riemann integrable on $[0,1]$."

True ______ *False* ______