Math 602 Exam 1 (02/08/11). Solutions

1. Give an example of a strictly increasing function $\alpha\colon [0,1] \to \mathbb{R}$ such that $\int_0^1 x\, d\alpha(x) = 5$ and $\int_0^1 x^2\, d\alpha(x) = 2$.

Construction. Recall that $\int f\, d\alpha$ is linear with respect to α; it behaves well under addition or rescaling of α. Once we find α for which the integrals given above have ratio $5\colon 2$, we can multiply α by a constant to get the desired results.

The obvious candidate $\alpha(x) = x$ gives ratio $\frac{1}{2}\!:\!\frac{1}{3}$ (or 3:2). We need to increase the first integral compared to the second one.

For this we should put more "mass" closer to 0, where x is much bigger than x^2. One way to do this is to add a point mass m at $c \in (0,1)$, both numbers to be chosen later. This means replacing $\alpha(x) = x$ with the following function:

$$\alpha_1(x) = \begin{cases} x & \text{if } x < c \\ x+m & \text{if } x \geqslant c \end{cases} \tag{1}$$

(Note that α_1 is α plus a step function.) Computation yields

$$\int_0^1 x\, d\alpha_1(x) = mc + \frac{1}{2}, \qquad \int_0^1 x^2\, d\alpha_1(x) = mc^2 + \frac{1}{3}.$$

We want $mc = 9/2$ and $mc^2 = 5/3$. This is achieved with $c = 10/27$ and $m = 20/81$. With this choice, function (1) satisfies the requirements. □

Instead of adding a point mass, one can make the density different on two halves of the interval: $\alpha_2' = a$ on $[0, 1/2]$ and $\alpha_2' = b$ on $[1/2, 1]$, with constants to be chosen later. This gives

$$\int_0^1 x\, d\alpha_2(x) = \frac{a}{8} + \frac{3b}{8} \qquad \int_0^1 x^2\, d\alpha_1(x) = \frac{a}{24} + \frac{7b}{24}$$

It remains to solve two linear equations to find that $a = 34$ and $b = 2$ work. Thus,

$$\alpha_2(x) = \begin{cases} 34x & \text{if } x < 1/2 \\ 2x+16 & \text{if } x \geqslant 1/2 \end{cases} \tag{2}$$

is also a solution. □

2. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = (-1)^{\lfloor x^2 \rfloor}$ where $\lfloor x^2 \rfloor$ is the greatest integer that does not exceed x^2.

Prove that $\lim\limits_{T\to\infty} \int_0^T f\, dx$ exists (as a real number).

Proof. Another way to express f is by saying that $f(x) = (-1)^k$ when $k \leqslant x^2 < k+1$, for $k = 0, 1, 2, \dots$. Hence

$$\int_{\sqrt{k}}^{\sqrt{k+1}} f\, dx = (-1)^k \left(\sqrt{k+1} - \sqrt{k}\right)$$

(although f is different from $(-1)^k$ at the right endpoint, this does not change the integral, as in Homework 2). It follows that

$$\int_0^{\sqrt{n}} f\,dx = \sum_{k=0}^{n-1}(-1)^k\left(\sqrt{k+1}-\sqrt{k}\right) = \sum_{k=0}^{n-1}\frac{(-1)^k}{\sqrt{k+1}+\sqrt{k}}.$$

This sum has a limit as $n\to\infty$, because the series converges (alternating test).

Given any $T>0$, let $n=\lfloor T^2\rfloor$ and write

$$\int_0^T f\,dx = \int_0^{\sqrt{n}} f\,dx + \int_{\sqrt{n}}^T f\,dx$$

where the absolute value of the last integral does not exceed

$$T-\sqrt{n} = \frac{T^2-n}{T+\sqrt{n}} \leqslant \frac{1}{T+\sqrt{n}} \to 0 \text{ as } n\to\infty.$$

Hence $\int_0^T f\,dx$ has the limit $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{k+1}+\sqrt{k}}$. □

3. Suppose that $f\colon[0,1]\to\mathbb{R}$ is integrable ($f\in\mathscr{R}$) and the set $\{x\in[0,1]\colon f(x)\neq 0\}$ is at most countable.

Prove that $\displaystyle\int_0^1 f\,dx = 0$.

Proof. Since $\displaystyle\left|\int_0^1 f\,dx\right| \leqslant \int_0^1 |f|\,dx$ (the integral triangle inequality), it suffices to prove that $\int_0^1|f|\,dx=0$. Any lower sum of this integral is 0, because every subinterval contains points where $|f|=0$. Hence the supremum of lower sums is also 0. Since f is integrable, so is $|f|$. We conclude that $\int_0^1|f|\,dx=0$. □

One can do without the triangle inequality by observing that for any subinterval I in any partition $\sup_I f\geqslant 0$, hence $U(P,f)\geqslant 0$. Since 0 is a lower bound for all upper sums, we conclude that $\overline{\int_0^1} f\geqslant 0$. A similar argument leads from $\inf_I f\leqslant 0$ to $\underline{\int_0^1} f\leqslant 0$. But the condition $f\in\mathscr{R}$ means that the upper and lower integrals have the same value, and this common value can only be 0. □

4. Let $\gamma\colon[0,1]\to\mathbb{R}^2$ be a rectifiable curve such that γ attains the values $(0,0)$, $(1,0)$, and $(0,1)$, not necessarily in this order.

Prove that the length of γ is at least 2.

Proof. Let $a<b<c$ be some points in $[0,1]$ where γ attains the aforementioned three values (in some order). Let P be the partition formed by these points together with 0 and 1. Then

$$\Lambda(\gamma)\geqslant\Lambda(P,\gamma)\geqslant|\gamma(b)-\gamma(a)|+|\gamma(c)-\gamma(b)|\geqslant 1+1=2$$

because the distance between any two of the points $(0,0)$, $(1,0)$, and $(0,1)$ is at least 1. □

5. Let $\alpha\colon [0,1] \to \mathbb{R}$ be an increasing function. Suppose that $f \in \mathscr{R}(\alpha)$ on $[0,1]$. Prove that $f \in \mathscr{R}(\beta)$ on $[0,1]$, where $\beta = \alpha^3$.

Proof. We know that for any $\epsilon > 0$ there is P such that

$$\sum_{i=1}^{n} \operatorname*{osc}_{[x_{i-1},x_i]} f(\alpha(x_i) - \alpha(x_{i-1}) < \epsilon.$$

We want to estimate a similar sum with β. Let $M = \max(|\alpha(0)|, |\alpha(1)|)$. The function $\varphi(t) = t^3$ has derivative $\varphi'(t) = 3t^2$, hence $|\varphi'(t)| \leqslant 3M^2$ on $[-M, M]$. By the Mean Value Theorem $|t^3 - s^3| \leqslant 3M^2|t - s|$ for $t, s \in [-M, M]$. (This can be also proved algebraically, using $t^3 - s^3 = (t - s)(t^2 + ts + s^2)$.) Thus,

$$\sum_{i=1}^{n} \operatorname*{osc}_{[x_{i-1},x_i]} f(\beta(x_i) - \beta(x_{i-1}) \leqslant 3M^2 \sum_{i=1}^{n} \operatorname*{osc}_{[x_{i-1},x_i]} f(\alpha(x_i) - \alpha(x_{i-1}) < 3M^2\epsilon$$

Since ϵ can be chosen arbitrarily small, $f \in \mathscr{R}(\beta)$.

6. "If $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at every point of $\mathbb{R}$, then for any $a < b$ we have $\int_a^b f'(x)\,dx = f(b) - f(a)$."

False: the derivative of a differentiable function is not necessarily integrable.

7. "If $\mathbf{f}\colon [0,2] \to \mathbb{R}^2$ is a continuous vector-valued function such that $3 \leqslant |\mathbf{f}(x)| \leqslant 5$ for every $x \in [0,2]$, then $6 \leqslant \left|\int_0^2 \mathbf{f}(x)\,dx\right| \leqslant 10$."

False. If f winds around the origin (say, in a circle of radius 4), the integral may even be equal to 0. For a concrete example, take $\mathbf{f}(t) = (4\cos \pi t, 4 \sin \pi t)$.