MATH 601 FINAL EXAM SOLUTIONS.

1. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a function such that $f(0) = f'(0) = 0$ and $f''(x) \geqslant 10$ for all $x \in \mathbb{R}$. Prove that $f(2) \geqslant 20$.
Proof. Let P_1 be the Taylor polynomial of degree (at most) 1 with center 0; this polynomial is identically 0. Apply Taylor's theorem:

$$f(2) = P_1(2) + \frac{f''(\xi)}{2} 2^2 = 2f''(\xi) \geqslant 20$$

2. Suppose $\{x_n \colon n = 1, 2, \dots\}$ is a sequence in a complete metric space X such that the series

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1})$$

converges. Prove that $\lim_{n\to\infty} x_n$ exists.
Proof. Since X is complete, it suffices to show that $\{x_n\}$ is a Cauchy sequence: that is, for any $\varepsilon > 0$ there exists N such that $d(x_n, x_m) < \varepsilon$ whenever $m, n \geqslant N$. We may assume $m < n$ without losing generality. By the (generalized) triangle inequality,

$$d(x_m, x_n) \leqslant \sum_{k=m}^{n-1} d(x_k, x_{k+1}) \tag{1}$$

The Cauchy criterion for convergence of a series tells us that for any $\varepsilon > 0$ there is N such that the right-hand side of (1) is less than ε whenever $m \geqslant N$. Hence the other side is less than ε too.

3. Let a and b be distinct points of a connected metric space X. Let r be a number such that $0 < r < d(a, b)$. Prove that there exists $x \in X$ such that $d(x, a) = r$.
Proof. By contrapositive. Suppose there is no such x. Then $X = A \cup B$ where $A = \{x \in X \colon d(x, a) < r\}$ and $B = \{x \in X \colon d(x, a) > r\}$ are disjoint sets. Since $a \in A$ and $b \in B$, neither set is empty. It remains to show that A and B are separated. For this it suffices to show that they are both closed. It is easier to show that complements (which are B and A again) are open. Indeed, if $x \in A$, then let $\delta = r - d(x, a)$ and use the triangle inequality to confirm that $N_\delta(x) \subset A$. And if $x \in B$, then let $\delta = d(x, a) - r$ and use the triangle inequality to confirm that $N_\delta(x) \subset B$.

4. Suppose that $f\colon (0, +\infty) \to \mathbb{R}$ is a uniformly continuous function such that $f(2x) = f(x)$ for all $x \in (0, \infty)$. Prove that f is a constant function.
Proof. Suppose that $f(a) \neq f(b)$ for some $a, b \in \mathbb{R}$. Let $\varepsilon = \frac{|f(a)-f(b)|}{2}$. By uniform continuity, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever} \quad |x - y| < \delta. \tag{2}$$

Choose a positive integer k large enough so that

$$\frac{|a-b|}{2^k} < \delta$$

Note that $f(a/2^k) = f(a)$ and $f(b/2^k) = f(b)$. Thus, we obtain a contradiction with (2) by choosing $x = {}^{a}\!/_{2^k}$ and $y = {}^{b}\!/_{2^k}$.

5. The power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 3. The radius of convergence of $\sum_{n=0}^{\infty} b_n z^n$ is equal to 4. Prove that the radius of convergence of $\sum_{n=0}^{\infty} (a_n b_n) z^n$ is at least 12.

Proof. We are given that $\limsup |a_n|^{1/n} = 1/3$ and $\limsup |b_n|^{1/n} = 1/4$. Thus, for any $\varepsilon > 0$ there exist integers N_1, N_2 such that

$$|a_n|^{1/n} < \frac{\sqrt{1+\varepsilon}}{3} \quad \text{whenever} \quad n \geqslant N_1$$
$$|b_n|^{1/n} < \frac{\sqrt{1+\varepsilon}}{4} \quad \text{whenever} \quad n \geqslant N_2$$

(This is a characteristic property of lim sup, Theorem 3.17.)
Thus, for $n \geqslant \max(N_1, N_2)$ we have

$$|a_n b_n|^{1/n} < \frac{1+\varepsilon}{12}$$

It follows that $\limsup |a_n b_n|^{1/n} \leqslant \frac{1+\varepsilon}{12}$. And since $\varepsilon > 0$ was arbitrary, $\limsup |a_n b_n|^{1/n} \leqslant 1/12$.

6. Suppose that $f\colon [0,1] \to \mathbb{R}$ is a continuous function. Define $g\colon [0,1] \to \mathbb{R}$ as follows:

$$g(x) = \sup\{f(t)\colon 0 \leqslant t \leqslant x\}$$

Prove that g is continuous.

Proof. Note that f is uniformly continuous (this is not strictly necessary for the proof, but uniform continuity makes it simpler). So, for any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. We claim that $|f(x) - f(y)| \leqslant \varepsilon$ whenever $|x - y| < \delta$; this is enough for (uniform) continuity of g). Without loss of generality take $x > y$. Note that $g(x) \geqslant g(y)$ because $g(x)$ is the supremum of a larger set. In the opposite direction,

$$g(x) \leqslant \max(g(y), \sup\{f(t)\colon y < t \leqslant x\}) \tag{3}$$

because the number on the right is an upper bound for $\{f(t)\colon 0 \leqslant t \leqslant x\}$. Furthermore, $f(y) + \varepsilon$ is an upper bound for $\{f(t)\colon y < t \leqslant x\}$ because $f(t) - f(y) < \varepsilon$ for any such t. Thus we conclude with

$$g(x) \leqslant \max(g(y), f(y) + \varepsilon) \leqslant g(y) + \varepsilon$$

as desired.

7. "If $f\colon \mathbb{R} \to \mathbb{R}$ is a function such that the set $\{x \in \mathbb{R}\colon f(x) > 0\}$ is uncountable, then there exists a number $\varepsilon > 0$ such that the set $\{x \in \mathbb{R}\colon f(x) \geqslant \varepsilon\}$ is uncountable."
True, because

$$\{x \in \mathbb{R}\colon f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R}\colon f(x) > 1/n\}.$$

If all sets on the right were at most countable, the set on the left would be as well.

8. "If A and B are open subsets of $\mathbb{R}$, then the set $A \setminus B$ is also open."
False. Take $A = (0, 2)$ and $B = (0, 1)$; the difference is $[1, 2)$.

9. "If $f\colon \mathbb{R} \to \mathbb{R}$ is a function differentiable at 0, then $\lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{\sqrt{|x|}} = 0$."
True, because

$$\lim_{x\to 0} \frac{f(x) - f(0)}{\sqrt{|x|}} = \lim_{x\to 0} \frac{f(x) - f(0)}{x} \cdot \frac{x}{\sqrt{|x|}} = f'(0) \cdot \lim_{x\to 0} \frac{x}{\sqrt{|x|}} = f'(0) \cdot 0 = 0$$