MATH 601 FINAL EXAM (12/17/10). MAX TOTAL SCORE 50.

YOUR NAME: ____

READ THIS FIRST: Do not open the exam booklet until told to do so. Out of the first **six** problems, do any **four** (worth 10 points each). If you attempt more than four problems, indicate which ones are not to be graded.

The second part consists of **three** True/False questions, out of which you should answer exactly **two** (worth 5 points each).

You may not use the textbook or notes. Rough work can be done on back pages of the booklet. Giving or receiving unauthorized aid during an exam is a violation of Syracuse University Academic Integrity Policy.

Part I: Do four out of six problems. If you attempt more than four problems, indicate which ones are not to be graded. Support your claims.

1. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a function such that $f(0) = f'(0) = 0$ and $f''(x) \geqslant 10$ for all $x \in \mathbb{R}$. Prove that $f(2) \geqslant 20$.

2. Suppose $\{x_n\colon n = 1, 2, \dots\}$ is a sequence in a complete metric space X such that the series

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1})$$

converges. Prove that $\lim_{n\to\infty} x_n$ exists.

3. Let a and b be distinct points of a connected metric space X. Let r be a number such that $0 < r < d(a, b)$. Prove that there exists $x \in X$ such that $d(x, a) = r$.

4. Suppose that $f\colon (0,+\infty) \to \mathbb{R}$ is a uniformly continuous function such that $f(2x) = f(x)$ for all $x \in (0,\infty)$. Prove that f is a constant function.

5. The power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 3. The radius of convergence of $\sum_{n=0}^{\infty} b_n z^n$ is equal to 4. Prove that the radius of convergence of $\sum_{n=0}^{\infty} (a_n b_n) z^n$ is at least 12.

6. Suppose that $f\colon [0,1] \to \mathbb{R}$ is a continuous function. Define $g\colon [0,1] \to \mathbb{R}$ as follows:

$$g(x) = \sup\{f(t)\colon 0 \leqslant t \leqslant x\}$$

Prove that g is continuous.

Part II: Answer two out of three True/False questions. If you try more than two, indicate which one is not to be graded. You do not need to support your claims in this part.

7. "If $f\colon \mathbb{R} \to \mathbb{R}$ is a function such that the set $\{x \in \mathbb{R}\colon f(x) > 0\}$ is uncountable, then there exists a number $\varepsilon > 0$ such that the set $\{x \in \mathbb{R}\colon f(x) \geqslant \varepsilon\}$ is uncountable."

True ______ *False* ______

8. "If A and B are open subsets of $\mathbb{R}$, then the set $A \setminus B$ is also open."

True ______ *False* ______

9. "If $f\colon \mathbb{R} \to \mathbb{R}$ is a function differentiable at 0, then $\lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{\sqrt{|x|}} = 0$."

True ______ *False* ______