Math 601 Final Exam (12/21/09). Max total score 50.

Your name: ____

READ THIS FIRST: Do not open the exam booklet until told to do so. Out of the first **six** problems, do any **four** (worth 10 points each). If you attempt more than four problems, indicate which ones are not to be graded. The exam concludes with two True/False questions worth 5 points each. You may not use the textbook or notes. Rough work can be done on back pages of the booklet. Giving or receiving unauthorized aid during an exam is a violation of Syracuse University Academic Integrity Policy.

Part I: Do four out of six problems. If you attempt more than four problems, indicate which ones are not to be graded. Support your claims.

1. Let X be a metric space. Suppose that (x_n) is a sequence of elements of X such that $d(x_n, x_m) \geqslant 1$ whenever $n \neq m$. Prove that X is not compact.

2. Suppose that (x_n) is a Cauchy sequence in $\mathbb{R}$, and $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function. Prove that $(f(x_n))$ is a Cauchy sequence.

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be two differentiable functions. You are given that $f'(x) = g'(x)$ for all $x \neq 0$. Prove that $f'(0) = g'(0)$.

4. Let E be a closed subset of a metric space X. Prove that there exists a sequence of open sets $G_n \subset X$ such that (i) $G_{n+1} \subset G_n$ for all n; (ii) $\bigcap_{n=1}^{\infty} G_n = E$.

5. Given two power series: $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R_1, and $\sum_{n=0}^{\infty} b_n z^n$ has radius of convergence R_2. Suppose that $0 < R_1 < R_2 < \infty$. Prove that the radius of convergence of the power series $\sum_{n=0}^{\infty} (a_n + b_n) z^n$ is equal to R_1.

6. Let Y be a metric space. Suppose that $f\colon \mathbb{R} \to Y$ is a function such that for any $x \in \mathbb{R}$

$$\lim_{t \to x} \frac{d(f(t), f(x))}{|t - x|} = 0$$

Prove that f is a constant function; that is, $f(x) = f(0)$ for all $x \in \mathbb{R}$.

Part II: True/False questions, 5 points each. You do not need to support your claims in this part.

5. "If a, b, c are complex numbers such that $|a-b| = |b-c| = |a-c|$, then there exists a complex number d such that $|a-d| = |b-d| = |c-d|$."

True ______ *False* ______

6. "If a set $E \subset \mathbb{R}$ is such that $E \cap [-n, n]$ is at most countable for every $n \in \mathbb{N}$, then E is at most countable."

True ______ *False* ______