LOGISTIC EQUATION

The logistic equation is

$$p' = rp(1 - p/K) \tag{1}$$

where r and K are positive constants: r determines how fast the population is growing, K is the carrying capacity (the maximal population that the environment can support).

Since (1) is autonomous, we should look for equilibrium solutions first. The right hand side turns into 0 for two values of p, namely $p = 0$ and $p = K$. These are two equilibrium solutions.

To find other solutions, separate the variables and integrate.

$$\int \frac{K}{p(K-p)}\, dp = \int r\, dt \tag{2}$$

We need partial fraction decomposition to handle the integral on the left of (2). Its general form is

$$\frac{K}{p(K-p)} = \frac{A}{p} + \frac{B}{K-p} \tag{3}$$

where the numerators A and B are to be found. Multiply by the common denominator:

$$K = A(K-p) + Bp \tag{4}$$

Evaluation of both sides of (4) at $p = 0$ yields $K = AK$, hence $A = 1$.
Evaluation of both sides of (4) at $p = K$ yields $K = BK$, hence $B = 1$. Neat.

We use the decomposition (3) to integrate (2). The result is

$$\ln|p| - \ln|K-p| = rt + C$$

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Now some algebra:

$$\begin{aligned} \ln \frac{|p|}{|K-p|} &= rt + C \\ \frac{|p|}{|K-p|} &= e^C e^{rt} \\ \frac{p}{K-p} &= \pm e^C e^{rt} = C_1 e^{rt} \end{aligned}$$

Find C_1 by plugging in $t = 0$ and the initial population $p = p_0$:

$$\frac{p_0}{K-p_0} = C_1.$$

Back to algebraic manipulations:

$$\begin{aligned} \frac{p}{K-p} &= \frac{p_0}{K-p_0} e^{rt} \\ \frac{K-p}{p} &= \frac{K-p_0}{p_0} e^{-rt} \\ \frac{K}{p} - 1 &= \frac{K-p_0}{p_0} e^{-rt} \\ \frac{K}{p} &= 1 + \frac{K-p_0}{p_0} e^{-rt} \\ p &= \frac{K}{1 + \frac{K-p_0}{p_0} e^{-rt}} \end{aligned}$$

And so, at long last, we have the solution

$$\boxed{p = \frac{Kp_0}{p_0 + (K-p_0)e^{-rt}}} \tag{5}$$

We should not forget the equilibrium solutions $p = 0$ and $p = K$. Fortunately, the formula (5) already contains both of them as special cases $p_0 = 0$ and $p_0 = K$.

In class the solution (5) was written in a different, but equivalent, way:

$$p = \frac{p_0 e^{rt}}{1 + \frac{p_0}{K}(e^{rt} - 1)} \tag{6}$$