MAT 414 PROJECT 2: COMPARISON OF TWO NUMERICAL METHODS

In this project you will compare the performance of two numerical methods:

- the Euler method, which you used in Project 1
- the RK2 method (abbreviation of "the Runge-Kutta method of second order").

The performance of the numerical methods will be compared on the equation

$$y' = ry \quad \text{with initial condition } \; y(0) = y_0$$

Here $r = 1 + 0.1 *$ first digit of your SUID, and $y_0 = 10 +$ second digit of your SUID.

For example, my SUID begins with 79, which means that for me $r = 1.7$ and $y_0 = 19$. Therefore, I will work with the equation $y' = 1.7y$ and initial condition $y(0) = 19$. This equation is easy to solve exactly, and we already know the solution: $y = y_0 e^{rt}$, which in my case is $y = 19e^{1.7t}$.

GOALS OF THE PROJECT

(1) Find the exact values of y at $t = 0.05, 0.10, 0.15, \dots, 1.95, 2$ by solving the equation as above
(2) Find approximate values of y at $t = 0.05, 0.10, 0.15, \dots, 1.95, 2$ using the Euler method
(3) Find approximate values of y at $t = 0.05, 0.10, 0.15, \dots, 1.95, 2$ using the RK2 method
(4) Compare the accuracy of the Euler and RK2 methods.

THE EULER METHOD

Since $y(0)$ is known, we can find $y'(0)$ from the equation. Then we calculate

$$y(0.05) \approx y(0) + 0.05\, y'(0)$$

This is reasonable because $y'(0)$ is the rate of change of y at time 0, so we can expect y to change by approximately $0.05\, y'(0)$ over the interval from 0 to 0.05.

Now that we have $y(0.05)$, the process repeats. We find $y'(0.05)$ from the equation, and then

$$y(0.10) \approx y(0.05) + 0.05\, y'(0.05)$$

Again, this is reasonable because $y'(0.05)$ is the rate of change of y at time 0.05, and we expect y approximately $0.05\, y'(0.05)$ over the interval from 0.05 to 0.10.

THE RK2 METHOD

The Euler's method approximates the change of y over the interval such as $[0, 0.05]$ using the value of y' only at the left endpoint of this interval. The RK2 method uses the average of y' at both endpoints. Thus, the basic idea is to take

$$y(0.05) \approx y(0) + 0.05\, \frac{y'(0) + y'(0.05)}{2}$$

But to use this formula, we need to find $y'(0.05)$, and for that we need $y(0.05)$...

(over)

The idea is to tentatively **predict** $y(0.05)$ using Euler's methid

$$y(0.05) \approx y(0) + 0.05\, y'(0)$$

and use this prediction to calculate $y'(0.05)$. Then we take

$$y(0.05) \approx y(0) + 0.05\, \frac{y'(0) + y'(0.05)}{2}$$

correcting the original prediction. Then the process repeats.

For example, with my numbers I would have $y'(0) = 1.7 \cdot 19 = 32.3$, which predicts $y(0.05) \approx 19 + 0.05 \cdot 32.3 \approx \mathbf{20.62}$. On the basis of this prediction, $y'(0.05) \approx 1.7 \cdot 20.62 = 35.05$. Using the average $\frac{y'(0) + y'(0.05)}{2} \approx 33.67$ instead of just $y'(0)$, I arrive at a more precise value for $y(0.05)$, namely $19 + 0.05 \cdot 33.67 \approx \mathbf{20.68}$. (The exact solution gives $y(0.05) = 19e^{1.7 \cdot 0.05} \approx \mathbf{20.6856}\ldots$)

Technical details

I recommend using spreadsheet software, as for Project 1.

To begin, download the template at goo.gl/Q2XcX (or make your own copy in Google Drive).

Column A contains all necessary values of t. Columns B is for the exact solution. For example, my cell B3 would have

```
=19*EXP(1.7*A3)
```

Columns D and E implement the Euler method, just as in Project 1. In Column F you should calculate the error of this method, which is the absolute value of the difference between exact values in column B and approximation in column D. Like this:

```
=ABS(B3-D3)
```

Columns H through L implement the RK2 method. For example, cell H3 contains the initial value $y(0)$ (mine is 19, you should use yours). Cell I3 calculates $y'(0)$ according to the differential equation. Then cell J3 makes a prediction of $y(0.05)$:

```
=H3+I3*0.05
```

Cell K3 calculates $y'(0.05)$ according to the differential equation, using the prediction in cell J3. Finally, L3 takes the average of two values of the derivative

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=(I3+K3)/2
```

and this is what we use to move to the next row, by filling H4 with the formula

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=H3+L3*0.05
```

In column M you should calculate the error of the RK2 method, which is the absolute value of the difference between exact values in column B and approximation in column H.

Finally, answer the questions in Column O.

Submit your Project 2 spreadsheet on Blackboard by the end of **Wednesday, February 13**.